Wadge and Lipschitz Games

Definition 1. Let $\varphi : S \to T$ be a function between non-empty pruned Trees (no terminal nodes).

- φ is called *monotone*, if $s \subseteq s' \Rightarrow \varphi(s) \subseteq \varphi(s')$ and *coontinuous*, if it is monotone and for all $x \in [S]$: $\lim_{n\to\infty} \varphi(x \upharpoonright n) = \infty$.
- φ ist called *Lipschitz*, if it is monotone and for all $s \in S$: $lh(s) = lh(\varphi(s))$. φ is called a *contraction* if $lh(\varphi(s)) = lh(s) + 1$.
- For monotone functions φ, let $D_\varphi = \{x \in [S] \mid \lim_{n\to\infty} lh(\varphi(x \upharpoonright n)) = \infty\}$ and $f_\varphi : D_\varphi \to [T]$ be defined as $f_\varphi(x) = \bigcup_{n\in\omega} \varphi(x \upharpoonright n)$.

Obviously φ is continuous iff $D_\varphi = [S]$.

Bemerkung 1. If φ is Lipschitz, then:

- For all $x \in [S]$ we have $\lim_{n\to\infty} lh(\varphi(x \upharpoonright n)) = \lim_{n\to\infty} lh(x \upharpoonright n) = \infty$ and thus φ is continuous.
- For all $x, y \in [S]$, if we have $x \upharpoonright n = y \upharpoonright n$, then clearly $f_\varphi(x) \upharpoonright n = f_\varphi(y) \upharpoonright n$. It follows, that $d(f_\varphi(x), f_\varphi(y)) \leq d(x, y)$, which makes f_φ a Lipschitz function (in the usual sense) with constant ≤ 1.

  Conersely, if $f : [S] \to [T]$ has Lipschitz constant ≤ 1, then for all $x, y \in [S]$ we have $d(f(x), f(y)) \leq d(x, y)$, which means
  $$x \upharpoonright n = y \upharpoonright n \Rightarrow f(x) \upharpoonright n = f(y) \upharpoonright n$$
  and therefore $f \upharpoonright S =: \varphi$ is well defined, monotone and Lipschitz in our sense.
- A Function $f : [S] \to [T]$ with Lipschitz constant $\leq \frac{1}{2}$ - that is, we have
  $$x \upharpoonright n = y \upharpoonright y \Rightarrow f(x) \upharpoonright (n + 1) = f(y) \upharpoonright (n + 1)$$
  is similarly induced by a contraction.

Definition 2. Let X be a topological space and $F \subseteq {}^X X$ a family of functions closed under composition and containing the identity and all constant functions. Let $A, B \subseteq X$, then A is *F-reducible* to B - $A \leq_F^X B$ - iff there is $f \in F$ such that $A = f^{-1}(B)$.

The conditions for F imply, that $\leq_F$ is reflexive and transitive. Set $A \equiv_F B \Leftrightarrow A \leq_F B \wedge B \leq_F A$. We call $[A]_F := \{B \mid B \equiv_F A\}$ the *F-degree* of A.
Note that $A \leq_F B \Leftrightarrow A^C \leq_F B^C$.
The *dual* of $[A]_F$ is $[A^C]_F$. A set A is called *F-self-dual* iff $A \equiv_F A^C$. The notion can be extended to F-degrees. $\leq_F$ is a partial order on F-degrees.
If $F \subseteq G$, then $\leq_G$ is *coarser* than $\leq_F$. We have:

$$A \leq_F B \Rightarrow A \leq_G B \qquad A \text{ is } F\text{-self-dual} \Leftrightarrow A \text{ is } G\text{-self-dual} \qquad [A]_F \subseteq [A]_G$$

Also, the dual to $[X]_F = \{X\}$ is $[\emptyset]_F = \{\emptyset\}$. Obviously for every set A we have $X \leq_F A$ and $\emptyset \leq_F A$ (both because we have constant functions).

Definition 3. The *Lipschitz game* $G_L^X(A, B)$ *for* $A, B \subseteq {}^\omega X$ is the game

I	a_0		a_1		...
II		b_0		b_1	...

where player II wins iff
$$a = (a_i)_{i\in\omega} \in A \Leftrightarrow b = (b_i)_{i\in\omega} \in B.$$

Theorem 1. *The Lipschitz game $G_L(A, B)$ is equivalent to a game $G(C)$. The payoff set C is Borel iff A and B are Borel.*

Proof. Define $A \sqcup B := \{(a_0, b_0, a_1, b_1, a_2, b_2, ...) \mid (a_i) \in A, (b_i) \in B\}$. Player I wins iff $a \in A^C$ and $b \in B$, or $a \in A$ and $b \in B^C$. That means $G_L(A, B) = G((A^C \sqcup B) \cup (A \sqcup B^C)) =: G(C)$.
For Borel: Observe, that $A \sqcup B$ is a basic set iff A, B are basic sets. Proceed with unions and intersections. □

Let L be the set of Lipschitz functions on ${}^\omega X$.

Theorem 2. 1. *II wins $G_L(A, B) \Leftrightarrow A \leq_L B$.*

2. *I wins $G_L(A, B) \Rightarrow B^C \leq_L A$.*

Proof. 1. A winning strategy for II is a function $\varphi : {}^{<\omega}X$ to ${}^{<\omega}X$ with $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$, $lh(s) = lh(\varphi(s))$ and: if there is $a \in A$ (resp. A^C) with $a \upharpoonright n = s$, then there is $b \in B$ (resp. B^C) with $b \upharpoonright n = \varphi(s)$.

Hence, a winning strategy exists iff there is a Lipschitz function $f_\varphi : {}^\omega X \to {}^\omega X$ with $f^{-1}(B) = A$, i.e. iff $A \leq_L B$.

2. A winning strategy for I on the other hand is a similar function φ with $lh(\varphi(s)) = lh(s) + 1$ and: if there is $b \in B^C$ (resp. B) with $b \upharpoonright n = s$, then there is $a \in A$ (resp. A^C) with $a \upharpoonright (n + 1) = \varphi(s)$.

Hence, a winning strategy yields a contraction f_φ with $f^{-1}(A) = B^C$, i.e. $B^C \leq_L A$.

□

Note, that the converse for 2. doesn't work, since a function witnessing $B^C \leq_L A$ need not necessarily be a contraction and thus doesn't necesserily represent a winning strategy.

Definition 4. The *Wadge game* $G_W^X(A,B)$ is a variant of the Lipschitz game, where player II is allowed to *pass*; i.e. he is allowed to play an Element $p \notin X$ which is disregarded when forming the sequence b. However, he has to move infinitely often.

Theorem 3. *The Wadge game $G_W(A,B)$ is equivalent to a game $G(C)$.*

Proof. Similar as above, but more complicated. □

Let W be the set of continuous maps on ${}^\omega X$.

Theorem 4. *1. II wins $G_W(A,B) \Leftrightarrow A \leq_W B$*

2. I wins $G_W(A,B) \Rightarrow B^C \leq_L A \Rightarrow B^C \leq_W A$.

Proof. 1. Assume $\tau : {}^{<\omega}X \to X \cup \{p\}$ a strategy for II. Define $\hat{\tau} : {}^{<\omega}X \to {}^{<\omega}X$ as the restriction to proper moves, i.e. recursively:

$$\hat{\tau}(s) = \begin{cases} \emptyset & s = \emptyset \\ \hat{\tau}(s \upharpoonright lh(s)-1)^\frown \tau(s) & \tau(s) \neq p \\ \hat{\tau}(s \upharpoonright lh(s)-1) & \textit{otherwise.} \end{cases}$$

Clearly, $\hat{\tau}$ is monotone, and since player II will play infinitely many elements from X, $\hat{\tau}$ is continuous. Conversely, for a continuous function φ we can define a strategy τ with $\hat{\tau} = \varphi$. Obviously, τ is a winning strategy iff $\hat{\tau}$ witnesses $A \leq_W B$.

2. Assume I has a winning strategy. Since II can pass (more or less) as often as he wants, this strategy yields a family of contractions witnessing $\neg B \leq_L A$.

□

Definition 5. The *semi-linear ordering principle* $\mathsf{SLO}^{\mathcal{F}}(X)$ for a set of (appropriate) functions $\mathcal{F}$ on ${}^\omega X$ is the statement:

$$\forall A, B \subseteq {}^\omega X : A \leq_{\mathcal{F}} B \text{ or } B^C \leq_{\mathcal{F}} A$$

Assume $\mathsf{SLO}^{\mathcal{F}}(X)$ holds. Then if A and B are incomparable, we get $A \equiv_{\mathcal{F}} B^C$ und thus for every $C \not\equiv_{\mathcal{F}} A, B$ we get

$$C <_{\mathcal{F}} A, B \text{ or } A, B <_{\mathcal{F}} C$$

Hence, the name *semi-linear*.

Wadge's Lemma. *Let $\mathcal{A}$ be a set of subsets of ${}^\omega\omega$ for which the Lipschitz-game is determined (e.g. the Borel sets). Then SLO^L (and thus SLO^W as well) holds on $\mathcal{A}$.*

Theorem 5. *Let $\mathcal{F} \supset L$ be a reducibility notion (e.g. $\mathcal{F} = W$).*

(a) $\mathsf{SLO}^L \Rightarrow \mathsf{SLO}^{\mathcal{F}}$ (for any set of subsets)

(b) If $\mathsf{SLO}^{\mathcal{F}}$ holds (for any set of subsets), then any self-dual $\mathcal{F}$-degree $[A]_{\mathcal{F}}$ is comparable with all other degrees.

(c) Assume SLO^L. If $[A]_{\mathcal{F}}$ is non-self-dual, then $[A]_{\mathcal{F}} = [A]_L$.

Let $A \oplus B := (0^\frown A) \cup (1^\frown B)$.

Proof. (a) clear

(b) clear

(c) Obviously $[A]_{\mathcal{F}} \supseteq [A]_L$. Assume $B \in [A]_{\mathcal{F}} \setminus [A]_L$. Then either $B <_L A$ or $A <_L B$: Assume A, B are incomparable, then $A \equiv_L B^C$ and thus $A \equiv_{\mathcal{F}} B^C$, contradiction to non-self-dual.

If $B <_L A$, then $B \oplus B^C \in [A]_{\mathcal{F}}$: Since $A \leq_{\mathcal{F}} B$ and $B \leq_{\mathcal{F}} B \oplus B^C$, it follows, that $A \leq_{\mathcal{F}} B \oplus B^C$. On the other hand, $A \not\leq_L B$ therefore $B^C, B \leq_L A$ and thus $B \oplus B^C \leq A$.

Analogously, if $A <_L B$, then $A \oplus A^C \in [A]_{\mathcal{F}}$. Since for any set $C \oplus C^C$ is self-dual, it follows, that A is $\mathcal{F}$-equivalent to a self-dual set, contradiction.

□