Wadge and Lipschitz Games

**Definition 1.** Let \( \varphi : S \to T \) be a function between non-empty pruned Trees (no terminal nodes).
- \( \varphi \) is called monotone, if \( s \subseteq s' \Rightarrow \varphi(s) \subseteq \varphi(s') \) and \( \text{continuous} \), if it is monotone and for all \( x \in [S] : \lim_{n \to \infty} \varphi(x \upharpoonright n) = \infty \).
- \( \varphi \) is called Lipschitz, if it is monotone and for all \( s \in S : lh(s) = lh(\varphi(s)) \). \( \varphi \) is called a contraction if \( lh(\varphi(s)) = lh(s) + 1 \).
- For monotone functions \( \varphi \), let \( D_\varphi = \{ x \in [S] | \lim_{n \to \infty} lh(\varphi(x \upharpoonright n)) = \infty \} \) and \( f_\varphi : D_\varphi \to [T] \) be defined as \( f_\varphi(x) = \bigcup_{n \in \omega} \varphi(x \upharpoonright n) \).

Obviously \( \varphi \) is continuous iff \( D_\varphi = [S] \).

**Bemerkung 1.** If \( \varphi \) is Lipschitz, then:
- For all \( x \in [S] \) we have \( \lim_{n \to \infty} lh(\varphi(x \upharpoonright n)) = \lim_{n \to \infty} lh(x \upharpoonright n) = \infty \) and thus \( \varphi \) is continuous.
- For all \( x, y \in [S] \), if we have \( x \upharpoonright n = y \upharpoonright n \), then clearly \( f_\varphi(x) \upharpoonright n = f_\varphi(y) \upharpoonright n \). It follows, that \( d(f_\varphi(x), f_\varphi(y)) \leq d(x, y) \), which makes \( f_\varphi \) a Lipschitz function (in the usual sense) with constant \( \leq 1 \).
- Conversely, if \( f : [S] \to [T] \) has Lipschitz constant \( \leq 1 \), then for all \( x, y \in [S] \) we have \( d(f(x), f(y)) \leq d(x, y) \), which means \( x \upharpoonright n = y \upharpoonright n \Rightarrow f(x) \upharpoonright n = f(y) \upharpoonright n \) and therefore \( f \upharpoonright S =: \varphi \) is well defined, monotone and Lipschitz in our sense.

- A Function \( f : [S] \to [T] \) with Lipschitz constant \( \leq \frac{1}{2} \) : that is, we have \( x \upharpoonright n = y \upharpoonright n \Rightarrow f(x) \upharpoonright (n+1) = f(y) \upharpoonright (n+1) \)

is similarly induced by a contraction.

**Definition 2.** Let \( X \) be a topological space and \( F \subseteq X \) a family of functions closed under composition and containing the identity and all constant functions. Let \( A, B \subseteq X \), then \( A \) is \( F \)-reducible to \( B \) - \( A \leq_F B \) - iff there is \( f \in F \) such that \( A = f^{-1}(B) \).

The conditions for \( F \) imply, that \( \leq_F \) is reflexive and transitive. Set \( A \equiv_F B \Leftrightarrow A \leq_F B \land B \leq_F A \). We call \( [A]_F := \{ B \mid B \equiv_F A \} \) the \( F \)-degree of \( A \).

Note that \( A \leq_F B \Leftrightarrow A \leq_B B \leq_F B \).

The dual of \( [A]_F \) is \( [A]^C \). A set \( A \) is called \( F \)-self-dual iff \( A \equiv_F A^C \). The notion can be extended to \( F \)-degrees.

\( \leq_F \) is a partial order on \( F \)-degrees.

If \( F \subseteq G \), then \( \leq_F \) is coarser than \( \leq_F \). We have:

\[ A \leq_F B \Rightarrow A \leq_G B \quad A \text{ is } F\text{-self-dual} \Rightarrow A \text{ is } G\text{-self-dual} \quad [A]_F \subseteq [A]_G \]

Also, the dual to \( [X]_F = \{ X \} \) is \( [\emptyset]_F = \{ \emptyset \} \). Obviously for every set \( A \) we have \( X \leq_F A \) and \( \emptyset \leq_F A \) (both because we have constant functions).

**Definition 3.** The Lipschitz game \( G^L(A, B) \) for \( A, B \subseteq \omega^\omega \) is the game

<table>
<thead>
<tr>
<th>I</th>
<th>a_0</th>
<th>a_1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>b_0</td>
<td>b_1</td>
<td>...</td>
</tr>
</tbody>
</table>

where player I wins iff

\[ a = (a_i)_{i \in \omega} \in A \Leftrightarrow b = (b_i)_{i \in \omega} \in B. \]

**Theorem 1.** The Lipschitz game \( G^L(A, B) \) is equivalent to a game \( G(C). \) The payoff set \( C \) is Borel iff \( A \) and \( B \) are Borel.

**Proof.** Define \( A \cup B := \{(a_0, b_0, a_1, b_1, a_2, b_2, ...) \mid (a_i) \in A, (b_i) \in B \} \). Player I wins iff \( a \in A^C \) and \( b \in B \), or \( a \in A \) and \( b \in B^C \). That means \( G^L(A, B) = G((A^C \cup B) \cup (A \cup B^C)) := G(C) \).

For Borel: Observe, that \( A \cup B \) is a basic set iff \( A, B \) are basic sets. Proceed with unions and intersections. \( \square \)

Let \( L \) be the set of Lipschitz functions on \( \omega^\omega \).

**Theorem 2.**

1. If wins \( G^L(A, B) \Rightarrow A \leq_L B \).
2. \( L \) wins \( G^L(A, B) \Rightarrow B^C \leq_L A \).

**Proof.**

1. A winning strategy for II is a function \( \varphi : \omega^\omega \to \omega^\omega \) with \( s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t), lh(s) = lh(\varphi(s)) \) and: if there is \( a \in A \) (resp. \( A^C \)) with \( a \upharpoonright n = s \), then there is \( b \in B \) (resp. \( B^C \)) with \( b \upharpoonright n = \varphi(s) \).

Hence, a winning strategy exists iff there is a Lipschitz function \( f_\varphi : \omega^\omega \to \omega^\omega \) with \( f^{-1}(B) = A \), i.e. iff \( A \leq_L B \).

2. A winning strategy for I on the other hand is a similar function \( \varphi \) with \( lh(\varphi(s)) = lh(s) + 1 \) and: if there is \( b \in B^C \) (resp. \( B \)) with \( b \upharpoonright n = s \), then there is \( a \in A \) (resp. \( A^C \)) with \( a \upharpoonright (n+1) = \varphi(s) \).

Hence, a winning strategy yields a contraction \( f_\varphi \) with \( f^{-1}(A) = B^C \), i.e. \( B^C \leq_L A \). \( \square \)

Note, that the converse for 2. doesn’t work, since a function witnessing \( B^C \leq_L A \) need not necessarily be a contraction and thus doesn’t necessarily represent a winning strategy.
Definition 4. The Wadge game \( G^X_W(A, B) \) is a variant of the Lipschitz game, where player II is allowed to pass; i.e. he is allowed to play an Element \( p \notin X \) which is disregarded when forming the sequence \( b \). However, he has to move infinitely often.

Theorem 3. The Wadge game \( G_W(A, B) \) is equivalent to a game \( G(C) \).

Proof. Similar as above, but more complicated.

Let \( W \) be the set of continuous maps on \( \omega X \).

Theorem 4. 1. \( \text{II wins } G_W(A, B) \iff A \leq_W B \)

2. \( \text{I wins } G_W(A, B) \Rightarrow B^C \leq_L A \Rightarrow B^C \leq_W A \).

Proof. 1. Assume \( \tau : \omega X \to X \cup \{p\} \) a strategy for II. Define \( \hat{\tau} : \omega X \to \omega X \) as the restriction to proper moves, i.e. recursively:

\[
\hat{\tau}(s) = \begin{cases} 
\emptyset & s = \emptyset \\
\hat{\tau}(s | lh(s) - 1) \tau(s) & \tau(s) \neq p \\
\hat{\tau}(s | lh(s) - 1) & \text{otherwise}.
\end{cases}
\]

Clearly, \( \hat{\tau} \) is monotone, and since player II can play infinitely many elements from \( X \), \( \hat{\tau} \) is continuous. Conversely, for a continuous function \( \varphi \) we can define a strategy \( \tau \) with \( \hat{\tau} = \varphi \). Obviously, \( \tau \) is a winning strategy if \( \tau \) witnesses \( A \leq_W B \).

2. Assume I has a winning strategy. Since II can pass (more or less) as often as he wants, this strategy yields a family of contradictions witnessing \( \neg B \leq_L A \).

Definition 5. The semi-linear ordering principle \( \text{SLO}^F(X) \) for a set of (appropriate) functions \( F \) on \( \omega X \) is the statement:

\[
\forall A, B \subseteq \omega X : A \leq_F B \text{ or } B^C \leq_F A
\]

Assume \( \text{SLO}^F(X) \) holds. Then if \( A \) and \( B \) are incomparable, we get \( A \equiv_F B^C \) and thus for every \( C \neq_F A, B \) we get \( C <_F A, B \) or \( A, B <_F C \).

Hence, the name semi-linear.

Wadge’s Lemma. Let \( A \) be a set of subsets of \( \omega \omega \) for which the Lipschitz-game is determined (e.g. the Borel sets). Then \( \text{SLO}^F \) (and thus \( \text{SLO}^W \) as well) holds on \( A \).

Theorem 5. Let \( F \supseteq L \) be a reducibility notion (e.g. \( F = W \)).

(a) \( \text{SLO}^L \Rightarrow \text{SLO}^F \) (for any set of subsets)

(b) If \( \text{SLO}^F \) holds (for any set of subsets), then any self-dual \( F \)-degree \( [A]_F \) is comparable with all other degrees.

(c) Assume \( \text{SLO}^L \). If \( [A]_F \) is non-self-dual, then \( [A]_F = [A]_L \).

Let \( A \oplus B := (A^c \cup B) \cup (1 - B) \).

Proof. (a) clear

(b) clear

(c) Obviously \( [A]_F \supseteq [A]_L \). Assume \( B \in [A]_F \setminus [A]_L \). Then either \( B \leq_L A \) or \( A \leq_L B \): Assume \( A, B \) are incomparable, then \( A \equiv_L B^C \) and thus \( A \equiv_F B^C \), contradiction to non-self-dual.

If \( B \leq_L A \), then \( B \oplus B^C \in [A]_F \). Since \( A \leq_F B \) and \( B \leq_F B \oplus B^C \), it follows that \( A \leq_F B \oplus B^C \). On the other hand, \( A \leq_L B \) therefore \( B^C \), \( B \leq_L A \) and thus \( B \oplus B^C \leq_L A \).

Analogously, if \( A \leq_L B \), then \( A \oplus A^C \in [A]_F \). Since for any set \( C \oplus C^C \) is self-dual, it follows, that \( A \) is \( F \)-equivalent to a self-dual set, contradiction.