Let's compare the three main calculi used in the lecture – natural deduction, tableaux and resolution.

1 Natural Deduction

As the name suggests, natural deduction calculi are designed to mimic "natural" proofs, as done in informal mathematical settings. They consist of an *introduction rule* and an *elimination rule* for every connective and quantifier, that are relatively easy to come up with, if you know the semantics of a connective. For example, if you know what the connective $\wedge$ "means", and you need an introduction rule (that introduces a new $\wedge$) and an elimination rule (that gets rid of a $\wedge$, it's rather "natural" that the rules will have to look like this:

$$\text{Introduction:}\frac{A \quad B}{A \wedge B}$$

$$\text{Elimination:}\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

The disadvantage is that natural deduction calculi need some kind of intuition/reasoning about what to do – in most cases, several rules can be applied, and a *choice* has to be made which one to apply. Consequently, they are relatively non-algorithmic.

We will prove

$$\exists X.(P(X) \Rightarrow \forall Y.P(Y))$$

...where $P \in \Sigma_1^p$ – i.e. P is a unary predicate symbol:

We start with the lemma $(\neg\forall Y.P(Y)) \Rightarrow \exists Y.\neg P(Y)$:

(Assumption)^1	$\neg\forall Y.P(Y)$
(Assumption)^2	$\neg\exists Y.\neg P(Y)$
(Assumption)^3	$\neg P(Y)$
$\exists$-Introduction	$\exists Y \neg P(Y)$
$\bot$-Introduction	$\bot$
$\neg$-introduction^3	$\neg\neg P(Y)$
$\neg$-Elimination	$P(Y)$
$\forall$-Introduction	$\forall Y.P(Y)$
$\bot$-Introduction	$\bot$
$\neg$-introduction^2	$\neg\neg\exists Y.\neg P(Y)$
$\neg$-Elimination	$\exists Y.\neg P(Y)$
$\Rightarrow$ -Introduction^1	$(\neg\forall Y.P(Y)) \Rightarrow \exists Y.\neg P(Y)$

Now for the actual proof, using the above lemma:

(TND)	$(\forall Y.P(Y)) \vee (\neg\forall Y.P(Y))$
(Assumption)^1	$\forall Y.P(Y)$
(Assumption)^2	$P(X)$
$\Rightarrow$ -Introduction^2	$P(X) \Rightarrow \forall Y.P(Y)$
$\exists$-Introduction	$\exists X.(P(X) \Rightarrow \forall Y.P(Y))$
(Assumption)^1	$\neg\forall Y.P(Y)$
(Lemma)	$(\neg\forall Y.P(Y)) \Rightarrow \exists Y.\neg P(Y)$
$\Rightarrow$ -Elimination	$\exists Y.\neg P(Y)$
$\exists$-Elimination	$\neg P(c)$
(Assumption)^2	$P(c)$
$\bot$-Introduction	$\bot$
$\bot$-Elimination	$\forall Y.P(Y)$
$\Rightarrow$ -Introduction^2	$P(c) \Rightarrow \forall Y.P(Y)$
$\exists$-Introduction	$\exists X.(P(X) \Rightarrow \forall Y.P(Y))$
$\vee$ -Elimination^1	$\exists X.(P(X) \Rightarrow \forall Y.P(Y))$

2 Tableaux

Tableaux proofs are always proofs by contradiction. They work by attempting to construct a *countermodel* of the formula. If that attempt succeeds, we not only know that the formula is *not* a tautology, we also know *why* it is not, by giving us an explicit model in which the formula is false. If the attempt fails[1], the formula has to be a tautology.

Every proposition in a tableaux calculus is labelled either *true* ($\cdot^T$) or *false* ($\cdot^F$), starting with the formula we are interested in, labelled as *false*, since we try to construct a model in which that formula is false.

Every rule in a tableaux calculus is guided by the question "what would have to be the case, if the current formula were labelled correctly?" Consider for example the rule for an implication labelled false $(A \Rightarrow B)^F$: If this implication were (provably) false, then A would have to be true and B would have to be false, so we add A^T and B^F. This often leads to *branching*; for example, with the labelled formula $(A \wedge B)^F$: If that formula should be false, then *either* A has to be false *or* B has to be false (or both) – so we branch into two sub-proofs with assumptions A^F and B^F, respectively.

We will prove the same formula as above:

We start with the assumption that our formula is false and attempt to derive a contradiction from that:

(1)	$\exists X.(P(X) \Rightarrow \forall Y.P(Y))^F$	
(2)	$P(V_X) \Rightarrow \forall Y.P(Y)^F$	(from 1)
(3)	$P(V_X)^T$	(from 2)
(4)	$\forall Y.P(Y)^F$	(from 2)
(5)	$P(c_Y)^F$	(from 4)
(6)	$\bot[c_Y/V_X]$	

We arrive at a contradiction, since we concluded that $P(c_Y)^F$ for some constant c_Y, but we also concluded $P(V_X)^T$, where V_X is a free variable, which we are allows to substitute by any term. So substituting V_X by c_Y yields both $P(c_Y)^F$ and $P(c_Y)^T$, contradiction.

3 Resolution

The resolution calculus is the most "computational" of the three, since it basically proceeds purely algorithmically. As tableaux, it is fundamentally guided by proofs by contradiction. The formula is first negated and deconstructed into a collection of *clauses*, which are subsequently resolved, yielding new clauses. A clause corresponds to a *disjunction*, all clauses are assumed to be true simultaneously. A set of clauses therefore corresponds to a conjunction of disjunctions – i.e. a formula in conjunctive normal form.

An empty clause corresponds to an empty disjunction, something that can not be satisfied – i.e. a contradiction. If we manage to derive an empty clause, we have proven that our assumption that the (negated) starting formula is true leads to a contradiction.

[1] *Fails* in the sense of: all calculus rules have been exhausted such that there provably is *no way* to construct a countermodel! All branches result in a contradiction.

Proof for

$$\forall X \forall Y \forall Z \exists U \exists V \exists W [(P(X,Y) \Rightarrow (P(Z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))]$$

...where $P \in \Sigma_2^p$, $R \in \Sigma_1^p$, and $a \in \Sigma_0^f$.

We first apply the CNF^1 calculus to derive a set of clauses:

$$\forall X \forall Y \forall Z \exists U \exists V \exists W [(P(X,Y) \Rightarrow (P(Z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))]^F$$

$$\exists U \exists V \exists W [(P(f_X, f_Y) \Rightarrow (P(f_Z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))]^F$$

$$[(P(f_X, f_Y) \Rightarrow (P(f_Z,a) \Rightarrow R(a))) \Rightarrow ((P(v_U, v_V) \wedge P(v_W, a)) \Rightarrow R(a))]^F$$

$$\{P(f_X, f_Y) \Rightarrow (P(f_Z,a) \Rightarrow R(a))^T\}; \{(P(v_U, v_V) \wedge P(v_W, a)) \Rightarrow R(a)^F\}$$

$$\{P(f_X, f_Y)^F, \quad P(f_Z,a) \Rightarrow R(a)^T\}; \{P(v_U, v_V) \wedge P(v_W, a)^T\}; \{R(a)^F\}$$

$$\{P(f_X, f_Y)^F, \quad P(f_Z,a)^F, \quad R(a)^T\}; \{P(v_U, v_V)^T\}; \{P(v_W, a)^T\}; \{R(a)^F\}$$

And now for the actual resolution (we only actually need the first two clauses):