Albert-Ludwigs-Universität Freiburg
Fakultät für Mathematik und Physik
Mathematisches Institut

The model completion for the theory of Heyting algebras

Bachelor Thesis

Dennis Müller
Matrikel-Nr. 3104758

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Advisor: Dr. Markus Junker

Contents

0 Introduction and Notation

The goal of this bachelor thesis is to prove the existence of a model completion for the theory of Heyting algebras (as outlined in [GZ97]). Heyting algebras can be interpreted as models of intuitionistic propositional logic (IpC); a fact that will be used extensively. The proof is based on the main theorem of [Pit92], which states that every second order propositional formula (with quantification over propositional variables) is intuitionistically equivalent to a first order formula. We can use this to show that the class of existentially closed Heyting algebras is an elementary class, proving that the theory has a model companion. The fact that Heyting algebras have the amalgamation property implies, that this model companion is in deed a model completion.

Some rudimentary knowledge of basic concepts in mathematical logic is expected; however, the prerequisites are attempted to be held at a reasonable minimum. Proofs taken from different authors are referenced as such; all others are my own work and should therefore be escpecially subject to critical consideration.

Intuitionistic logic and Pitts' Theorem are presented in section 1. In section 2, we will define lattices and Heyting algebras and explore their relation to IpC. In section 3 we will define model completions and present related model theoretic concepts and results, before we will take on the main proof in section 4.

In formulae (both in the propositional and predicate logical calculus), we will use the symbols $\neg, \forall, \exists, \wedge, \vee, \rightarrow, \top, \bot$ in that order of connective strength from strongest to weakest (e.g. $A \wedge B \rightarrow C$ is to be read as $(A \wedge B) \rightarrow C$). The symbols $\Rightarrow, \Leftrightarrow$ will be used for implication and equivalence of statements outside of formulae. In lattices and Heyting algebras we will use the symbols $\sqcap, \sqcup, \hookrightarrow$ for join, meet and the relative pseudocomplement respectively. Their connective strength is analogous to the logical symbols. The letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, ...$ will represent structures, whereas $A, B, C, ...$ will denote their underlying universes. If a formula φ holds in a model $\mathfrak{A}$ or a theory T, we will write $\mathfrak{A} \models \varphi$ or $T \models \varphi$. Substituting a variable x by some term or formula t in some formula φ will be written as $\varphi \left[\frac{t}{x}\right]$. We write $\varphi(x, \overline{y})$ to express that the formula φ contains the free variables x and $\overline{y}$, where $\overline{y}$ will denote a tuple of variables. Consequently, for a parameter or set of parameters $\overline{a}$, we will sometimes write $\varphi(\overline{a})$ for $\varphi(\overline{x}) \left[\frac{\overline{a}}{\overline{x}}\right]$, if it is clear which variables are supposed to be substituted. In propositional logic, every propositional variable is considered to be free. To avoid confusion, we denote equality within formulae by $\doteq$. We write $\mathfrak{A} \prec \mathfrak{B}$ to express that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$. $\mathfrak{A} \equiv \mathfrak{B}$ will denote elementary equivalence. For a subset $B \subseteq A$, the pair $(\mathfrak{A}, B)$ will denote the structure $\mathfrak{A}$ in the language extended by constant symbols from B.

All used notations are listed again in table 3.

1 Intuitionistic logic

Intuitionism as a philosophy goes back to the beginning of the twentieth century, particularly to L.E.J. Brouwer.[1] For him, all mathematical objects and properties were mental constructions and consequently, proofs ought to be similarly constructive. In particular, he considered proofs by contradiction (especially for the existence of objects with certain properties) to be unsatisfactory, and he rejected the law of the excluded middle, i.e. the proposition that “each particular mathematical problem can be solved in the sense that the question under consideration can either be affirmed, or refuted”.[2]

Intuitionistic logic was first fully formalized by Arend Heyting in 1928, even though Brouwer himself did not consider doing so useful in any way. Heyting also gave an intuitionistic formalization for arithmetic as well as for a kind of set theory (based on the notion of a “species”[3]). Several useful interpretations of intuitionistic logic have since been discovered, showing connections to the closure operator in topology, lattices and the lambda calculus[4], which has caused some interest among computer scientists.

1.1 Intuitionistic propositional logic (IpC)

The syntax in intuitionstic logic is the same as in classical logic. But unlike in classical logic, we will need all of the connectives $\neg, \wedge, \vee, \rightarrow$ in our axioms, as in IpC none of these is definable in terms of the others. We can however (as we will show) add the symbols $\top$ and $\bot$ (for “true” and “false”, respectively) and then replace $\neg\varphi$ with $\varphi \rightarrow \bot$.

We will use a Hilbert-type calculus, similar to the formalization given by Heyting[5], but clearer with regard to the rules of inference. We write $\vdash_I \varphi$ if the formula φ is derivable in IpC.

Definition 1.1 (IpC). [6] Intuitionistic propositional logic has the following axioms (for all formulae φ, ψ, χ):

IpC1 $\vdash_I \varphi \rightarrow (\psi \rightarrow \varphi)$

IpC2 $\vdash_I (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi))$

IpC3 $\vdash_I \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$

IpC4 $\vdash_I (\varphi \wedge \psi) \rightarrow \varphi$ $\quad\quad \vdash_I (\varphi \wedge \psi) \rightarrow \psi$

IpC5 $\vdash_I \varphi \rightarrow (\varphi \vee \psi)$ $\quad\quad \vdash_I \psi \rightarrow (\varphi \vee \psi)$

IpC6 $\vdash_I (\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$

[1] For more on the history of intuitionism see [VD86] or for intuitionism itself [Hey71].
[2] Attributed to Hilbert in [VD86, p.227]
[3] [Hey71, p.37]
[4] via the curry-howard isomorphism, see [SU98]
[5] [Hey71, p.105]
[6] This formalization is from [VD86, p.234]

IpC7 $\vdash_I (\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$

IpC8 $\vdash_I \varphi \to (\neg\varphi \to \psi)$

and the following rule of inference:

Modus Ponens (M.P.): If $\vdash_I \varphi$ and $\vdash_I \varphi \to \psi$, then $\vdash_I \psi$.

If we add $\vdash_I (\varphi \vee \neg\varphi)$ we get classical propositional logic. In fact, Gödel and Gentzen showed that a formula φ is provable in classical propositional logic, iff $\vdash_I \neg\neg\varphi$.[7] We call formulae that are provable directly from axioms *tautologies*.

Let Γ be a propositional theory (i.e. a set of formulae). We write $\Gamma \vdash_I \varphi$ if

- $\varphi \in \Gamma$ or
- φ is derivable from formulas in Γ and axioms using Modus Ponens.

Obviously, if $\Gamma' \vdash_I \varphi$ and $\Gamma' \subseteq \Gamma$, then $\Gamma \vdash_I \varphi$. We write $\psi \vdash_I \varphi$ for $\{\psi\} \vdash_I \varphi$. Also, it is useful to note that by this definition proofs in IpC are finite, which allows us to restrict ourselves in many cases to finite subsets of theories. We call two formulae φ, ψ *logically equivalent (in* Γ*)*, if $\Gamma \cup \{\varphi\} \vdash_I \psi$ and $\Gamma \cup \{\psi\} \vdash_I \varphi$ and denote this by writing $\Gamma \vdash_I \varphi \leftrightarrow \psi$. This notation will be justified by theorem 1.1, since it implies that φ and ψ are logically equivalent iff $\Gamma \vdash_I (\varphi \to \psi) \wedge (\psi \to \varphi)$. As usual in mathematical logic, we are (almost always) only interested in formulae up to logical equivalence.

Before we prove the deduction theorem, which will be of enormous utility, we will first prove the following statements:

Lemma 1.1.

(i) $\vdash_I (\varphi \wedge \psi)$ *iff* $\vdash_I (\psi \wedge \varphi)$

(ii) $\vdash_I (\varphi \wedge \psi) \wedge \chi$ *iff* $\vdash_I \varphi \wedge (\psi \wedge \chi)$

(iii) $\vdash_I \varphi \to \varphi$

Proof.

(i)

$\vdash_I \varphi \wedge \psi$	(1)
$\vdash_I (\varphi \wedge \psi) \to \varphi$	(2) (IpC4)
$\vdash_I (\varphi \wedge \psi) \to \psi$	(3) (IpC4)
$\vdash_I \varphi$	(4) (M.P. on (1) and (2))
$\vdash_I \psi$	(5) (M.P. on (1) and (3))
$\vdash_I \psi \to (\varphi \to (\psi \wedge \varphi))$	(6) (IpC3)
$\vdash_I \varphi \to (\psi \wedge \varphi)$	(7) (M.P. on (5) and (6))
$\vdash_I \psi \wedge \varphi$	(8) (M.P. on (4) and (7))

[7] [VD86, p.229]

(ii)

$\vdash_I (\varphi \wedge \psi) \wedge \chi$	(1)
$\vdash_I ((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge \psi)$	(2) (IpC4)
$\vdash_I ((\varphi \wedge \psi) \wedge \chi) \to \chi$	(3) (IpC4)
$\vdash_I (\varphi \wedge \psi)$	(4) (M.P. on (1) and (2))
$\vdash_I \chi$	(5) (M.P. on (1) and (3))
$\vdash_I (\varphi \wedge \psi) \to \varphi$	(6) (IpC4)
$\vdash_I (\varphi \wedge \psi) \to \psi$	(7) (IpC4)
$\vdash_I \varphi$	(8) (M.P. on (4) and (6))
$\vdash_I \psi$	(9) (M.P. on (4) and (7))
$\vdash_I \psi \to (\chi \to (\psi \wedge \chi))$	(10) (IpC3)
$\vdash_I \psi \wedge \chi$	(11) (M.P. on (9),(5) and (10))
$\vdash_I \varphi \to ((\psi \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi)))$	(12) (IpC3)
$\vdash_I \varphi \wedge (\psi \wedge \chi)$	(13) (M.P. on (8),(11) and (12))

The other direction can be proven analogously.

(iii)

$\vdash_I \varphi \to (\varphi \to \varphi)$	(1) (IpC1)
$\vdash_I (\varphi \to (\varphi \to \varphi)) \to ((\varphi \to ((\varphi \to \varphi) \to \varphi)) \to (\varphi \to \varphi))$	(2) (IpC2)
$\vdash_I (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to (\varphi \to \varphi)$	(3) (M.P. on (1) and (2))
$\vdash_I \varphi \to ((\varphi \to \varphi) \to \varphi)$	(4) (IpC1)
$\vdash_I \varphi \to \varphi$	(5) (M.P. on (4) and (3))

□

Remark 1. Commutativity and associativity now allow us to ignore parentheses and the order in conjunctive (sub-)formulae. Hence, they also justifiy writing formulae $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n$ simply as $\bigwedge_{i=1}^{n} \varphi_i$.

Theorem 1.1 (Deduction Theorem)**.** [8] *$\{\phi_1, ..., \phi_n\} \vdash_I \varphi$ iff $\{\phi_1, ..., \phi_{n-1}\} \vdash_I \phi_n \to \varphi$.*

Proof. The implication from right to left follows directly from Modus Ponens.

Let $\{\phi_1, ..., \phi_n\} \vdash_I \varphi$, then there is a finite sequence of formulae $S = (S_1, ..., S_m)$ such that $S_m = \varphi$ and each S_i is either one of the ϕ_j, an axiom or follows by Modus Ponens from two previous formulae. We will now define a sequence S^* inductively:

- If S_i is an axiom or one of the ϕ_j ($j \neq n$), we let $S_i^* = \phi_n \to S_i$, which is provable from S_i and IpC1 using M.P.

[8] The idea for this proof is well known.

- If $S_i = \phi_n$, we let $S_i^* = \phi_n \to \phi_n$, which is provable by Lemma 1.1.
- If S_i follows by Modus Ponens from S_j and $S_k = S_j \to S_i$ $(j, k < i)$, then by induction $S_j^* = \phi_n \to S_j$ and $S_k^* = \phi_n \to (S_j \to S_i)$ are provable from Γ. By IpC2 we have
$$(\phi_n \to S_j) \to ((\phi_n \to (S_j \to S_i)) \to (\phi_n \to S_i))$$
and by applying Modus Ponens twice we get $\phi_n \to S_i =: S_i^*$.

It follows, that every formula in S^* is provable from $\{\phi_1, ..., \phi_{n-1}\}$ and by definition we have $S_m^* = \phi_n \to \varphi$. □

Corollary 1.1. *$\Gamma \vdash_I \varphi$ iff there is a finite subset $\Gamma' \subset \Gamma$ such that $\vdash_I \bigwedge_{\phi \in \Gamma'} \phi \to \varphi$.*

Proof. First, note that if $\Gamma \vdash_I \varphi$, then there is a finite sequence of formulae proving φ, as in the previous proof. Hence, there is a finite subset $\Gamma' \subset \Gamma$ such, that $\Gamma' \vdash_I \varphi$ (Γ' contains only the formulae occuring in said finite sequence).

By remark 1 and using IpC3 and IpC4 repeatedly, we get
$$\Gamma' \vdash_I \bigwedge_{\phi \in \Gamma'} \phi \text{ and } \bigwedge_{\phi \in \Gamma'} \phi \vdash_i \psi \text{ for all } \psi \in \Gamma'$$
and hence
$$\Gamma' \vdash_I \varphi \text{ iff } \bigwedge_{\phi \in \Gamma'} \phi \vdash_I \varphi \text{ iff } \vdash_I \bigwedge_{\phi \in \Gamma'} \phi \to \varphi.$$
□

We can now define $\top := \varphi \to \varphi$ (or any other IpC-tautology, since they are logically equivalent) for any formula φ and $\bot := \neg\top$. We call a propositional theory Γ inconsistent, if $\Gamma \vdash_I \bot$. This will allow us to prove some further simple, but useful statements:

Lemma 1.2.

(i) $\vdash_I \varphi \to \neg\psi$ *iff* $\{\varphi, \psi\}$ *is inconsistent,*

(ii) $\vdash_I \neg\varphi$ *iff* $\vdash_I \varphi \to \bot$*,*

(iii) $\vdash_I \varphi \to \neg\neg\varphi$*,*

(iv) $\varphi \to \psi \vdash_I \neg\psi \to \neg\varphi$*,*

(v) $\vdash_I \neg\varphi$ *iff* $\vdash_I \neg\neg\neg\varphi$*.*

Remark 2. For (i), note that the same holds only in one direction if we take $\varphi \to \psi$ and $\{\varphi, \neg\psi\}$ (in general, $\{\varphi, \neg\psi\} \vdash_I \bot$ does *not* imply $\vdash_I \varphi \to \psi$). (ii) allows us to define negation in terms of $\bot$ and implication, which will be useful when we work with Heyting algebras later on. Also, note that the converse statements to (iii) and (iv) do *not* hold, which corresponds to rejecting (certain) proofs by contradiction.

Proof of lemma.

(i) Let $\vdash_I \varphi \to \neg\psi$, then by M.P. $\{\varphi, \psi\} \vdash_I \neg\psi$ and trivially $\{\varphi, \psi\} \vdash_I \psi$. By IpC8 we have $\vdash_I \psi \to (\neg\psi \to \bot)$ and by applying M.P. twice we get $\{\varphi, \psi\} \vdash_I \bot$.

For the converse, let $\{\varphi, \psi\} \vdash_I \bot$, then with the deduction theorem $\varphi \vdash_I \psi \to \neg\top$. By IpC1 we have $\vdash_I \top \to (\psi \to \top)$ and by IpC7 we have

$$\vdash_I (\psi \to \top) \to ((\psi \to \neg\top) \to \neg\psi).$$

Applying M.P. twice gives us $\varphi \vdash_I \neg\psi$ and thus $\vdash_I \varphi \to \neg\psi$.

(ii) Let $\vdash_I \neg\varphi$, then by IpC8 and M.P. we get $\varphi \vdash_I \bot$ and thus $\vdash_I \varphi \to \bot$.

Conversely, let $\vdash_I \varphi \to \bot$, then $\{\varphi, \top\}$ is inconsistent and thus by (i) we have $\vdash_I \top \to \neg\varphi$ and by M.P. $\vdash_I \neg\varphi$.

(iii) Follows from the fact that $\{\varphi, \neg\varphi\}$ is inconsistent and (i).

(iv) We have by IpC7

$$\vdash_I (\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$$

and by IpC1 $\neg\psi \vdash_I \varphi \to \neg\psi$. Hence, we get $\{\varphi \to \psi, \neg\psi\} \vdash_I \neg\varphi$ and thus $\varphi \to \psi \vdash_I \neg\psi \to \neg\varphi$.

(v) Implication from left to right follows from the fact that $\{\neg\varphi, \neg\neg\varphi\}$ is inconsistent. For the converse, if $\vdash_I \neg\neg\neg\varphi$, then $\{\top, \neg\neg\varphi\}$ is inconsistent. From (iii) we get that $\{\top, \varphi\}$ must be inconsistent as well and by (i) and M.P. we have $\vdash_I \neg\varphi$. □

Also, only one of De Morgan's Laws holds in IpC; the other one only holds for one direction:

Lemma 1.3.

(i) $\neg(\varphi \vee \psi) \vdash_I \neg\varphi \wedge \neg\psi$

(ii) $\neg\varphi \wedge \neg\psi \vdash_I \neg(\varphi \vee \psi)$

(iii) $\neg\varphi \vee \neg\psi \vdash_I \neg(\varphi \wedge \psi)$

Proof.

(i) We have $\vdash_I \varphi \to (\varphi \vee \psi)$ and thus $\vdash_I \neg(\varphi \vee \psi) \to \neg\varphi$. We can do the same with ψ and the claim follows.

(ii) We have $\neg\varphi \vdash_I \varphi \to \bot$ and $\neg\psi \vdash_I \psi \to \bot$. By IpC8

$$\vdash_I (\varphi \to \bot) \to ((\psi \to \bot) \to ((\varphi \vee \psi) \to \bot))$$

and hence $\neg\varphi \wedge \neg\psi \vdash_I (\varphi \vee \psi) \to \bot$.

(iii) By IpC6, we have

$$\vdash_I (\neg\varphi \to \bot) \to ((\neg\psi \to \bot) \to ((\neg\varphi \lor \neg\psi) \to \bot))$$

and by IpC8

$$\varphi \land \psi \vdash_I \neg\varphi \to \bot \text{ and } \varphi \land \psi \vdash_I \neg\psi \to \bot$$

and thus $\{\neg\varphi \lor \neg\psi, \varphi \land \psi\} \vdash_I \bot$. Therefore, the two formulae are inconsistent, which implies the claim. □

1.2 IpC^2 and Pitts' Theorem

Before we look at Pitts' Theorem (which lies at the heart of the proof for our main theorem) we need to introduce *second order intuitionistic propositional logic* (IpC^2), which is IpC extended by the usual quantifiers $\exists$ and $\forall$ ranging over propositional variables. The grammar of IpC^2 is consequently the usual one for propositional logic, extended by the following rule:

Given a propositional variable P and an IpC^2-formula φ, then $\forall P\varphi$ is an IpC^2-formula.

A sequent calculus for IpC^2 can be found in [Pit92], however an actual calculus is not necessary for our needs and thus omitted. The existential quantifier is then defined by

$$\exists P\varphi := \forall Q(\forall P(\varphi \to Q) \to Q),$$

where Q is any new propositional variable not occuring in any other formula under consideration. This definition might seem unusual at first, however the classical definition of the existential quantifier (i.e. $\exists P\varphi = \neg\forall P\neg\varphi$) would not work in IpC, since the introduction of negations would lead to problems. We will see in corollary 1.2, that the definition given here actually captures the intended meaning.

Given a first order propositional formula φ, we will denote by $\mathsf{Var}(\varphi)$ the set of all propositional variables occuring in φ. Analogously we define Var for propositional theories.

Andrew Pitts showed in [Pit92] that IpC^2 is already contained in IpC, in the sense that for every second order propositional formula there is a logically equivalent first order propositional formula. This result is trivial in classical propositional logic - since classical logic is two valued, the formula $\exists P\varphi$ is equivalent to $\varphi\left[\frac{\top}{P}\right] \lor \varphi\left[\frac{\bot}{P}\right]$ and $\forall P\varphi$ is equivalent to $\varphi\left[\frac{\top}{P}\right] \land \varphi\left[\frac{\bot}{P}\right]$.

Formally, what Pitts showed is the following:

Pitts' Theorem. [9] *Given a propositional variable P, for each first order intuitionistic proposition φ there is a first order intuitionistic proposition $\forall_P\varphi$ with $\mathsf{Var}(\forall_P\varphi) \subseteq \mathsf{Var}(\varphi) \setminus \{P\}$ and satisfying:*

(i) If $\Gamma \vdash_I \varphi$, then $\Gamma \vdash_I \forall_P\varphi$, provided $P \notin \mathsf{Var}(\Gamma)$ and

[9] [Pit92, Theorem 1]

(ii) If $\Gamma \vdash_I \forall_P \varphi$*, then for all* ψ*,* $\Gamma \vdash_I \varphi \left[\frac{\psi}{P}\right]$.

Remark 3. It is important to keep in mind that $\forall_P \varphi$ is a *first order* formula not to be confused with the second order formula $\forall P \varphi$. It might be helpful to think of $\forall_P$ as a unary function on the set of first order formulae.

The proof for Pitts' Theorem is purely proof-theoretical, but quite long and technical. Consequently, I leave out the details:

Proof (sketch). The proof is based on a specific form of a cut-free Gentzen-style sequent calculus labelled LJ^* given in table 1. A sequent has the form $\Delta \succ \varphi$ for a propositional formula φ and a finite *multiset* (i.e. a set, where each element is assigned a certain multiplicity) of formulae Δ. We then have $\Gamma \vdash_I \varphi$ iff there is a finite multiset Δ built up from formulae in Γ such that the sequent $\Delta \succ \varphi$ is provable in this calculus, denoted by $\vdash \Delta \succ \varphi$.

We define an order $<_{\mathsf{wt}}$ on formulae via the following *weight*-function:

- $\mathsf{wt}(\bot) = \mathsf{wt}(P) = 1$ for any propositional variable P,
- $\mathsf{wt}(\varphi \vee \psi) = \mathsf{wt}(\varphi \to \psi) = \mathsf{wt}(\varphi) + \mathsf{wt}(\psi) + 1$,
- $\mathsf{wt}(\varphi \wedge \psi) = \mathsf{wt}(\varphi) + \mathsf{wt}(\psi) + 2$.

We can then extend $<_{\mathsf{wt}}$ to a relation between finite multisets, such that:

$\Gamma <_{\mathsf{wt}} \Delta$ iff there are multisets $\Delta_1, \Delta_2, \Gamma'$, $\Delta_2 \neq \varnothing$ such that

- $\Delta = \Delta_1 \cup \Delta_2$ and $\Gamma = \Delta_1 \cup \Gamma'$,
- For all $\varphi \in \Gamma'$ there exists $\psi \in \Delta_2$ with $\varphi <_{\mathsf{wt}} \psi$.

Finally, we can extend $<_{\mathsf{wt}}$ to sequents by letting

$$(\Delta_1 \succ \varphi) <_{\mathsf{wt}} (\Delta_2 \succ \psi) \text{ if } (\Delta_1 \cup \{\varphi\}) <_{\mathsf{wt}} (\Delta_2 \cup \{\psi\}).^{10}$$

LJ^* is constructed such that the premise in each rule is always smaller with respect to $<_{\mathsf{wt}}$ than the conclusion (which is also the reason why we work with multisets instead of "normal" sets).

Now, given a multiset Δ, formula φ and propositional variable P, we define finite sets of formulae $\mathcal{E}_P(\Delta)$ and $\mathcal{A}_P(\Delta, \varphi)$ and corresponding formulae $E_P(\Delta) := \bigwedge_{\phi \in \mathcal{E}_P(\Delta)} \phi$ and $A_P(\Delta, \varphi) := \bigvee_{\phi \in \mathcal{A}_P(\Delta, \varphi)} \phi$ simultaniously via mutual recursion as in table 2.

We now have to show the following:

(i) $\mathsf{Var}(E_P(\Delta)) \subseteq \mathsf{Var}(\Delta) \setminus \{P\}$ and $\mathsf{Var}(A_P(\Delta, \varphi)) \subseteq \mathsf{Var}(\Delta \cup \{\varphi\}) \setminus \{P\}$

(ii) $\vdash \Delta \succ E_P(\Delta)$ and $\vdash \Delta \cup \{A_P(\Delta, \varphi)\} \succ \varphi$

[10] To clarify: $\Gamma \cup \{\psi\}$ means, the multiplicity of ψ in Γ is to be increased by 1. If $\psi \notin \Gamma, \psi$ is considered to have multiplicity 0. Analogously for the union of two arbitrary multisets – the mutliplicities are simply added.

$$\frac{}{\Gamma \cup \{P\} \succ P}\ (\text{Atom}) \qquad \frac{}{\Gamma \cup \{\bot\} \succ \varphi}\ (\bot \succ)$$

$$\frac{\Gamma \succ \varphi \quad \Gamma \succ \psi}{\Gamma \succ \varphi \wedge \psi}\ (\succ \wedge) \qquad \frac{\Gamma \cup \{\varphi, \psi\} \succ \chi}{\Gamma \cup \{\varphi \wedge \psi\} \succ \chi}\ (\wedge \succ)$$

$$\frac{\Gamma \succ \varphi}{\Gamma \succ \varphi \vee \psi}\ (\succ \vee_1) \qquad \frac{\Gamma \succ \psi}{\Gamma \succ \varphi \vee \psi}\ (\succ \vee_2)$$

$$\frac{\Gamma \cup \{\varphi\} \succ \chi \quad \Gamma \cup \{\psi\} \succ \chi}{\Gamma \cup \{\varphi \vee \psi\} \succ \chi}\ (\vee \succ) \qquad \frac{\Gamma \cup \{\varphi\} \succ \psi}{\Gamma \succ \varphi \to \psi}\ (\succ\to)$$

$$\frac{\Gamma \cup \{P, \varphi\} \succ \psi}{\Gamma \cup \{P, P \to \varphi\} \succ \psi}\ (\to\succ_1) \qquad \frac{\Gamma \cup \{\varphi \to (\phi \to \psi)\} \succ \chi}{\Gamma \cup \{(\varphi \wedge \phi) \to \psi\} \succ \chi}\ (\to\succ_2)$$

$$\frac{\Gamma \cup \{\varphi \to \phi, \psi \to \phi\} \succ \chi}{\Gamma \cup \{(\varphi \vee \psi) \to \phi\} \succ \chi}\ (\to\succ_3) \qquad \frac{\Gamma \cup \{\phi \to \psi\} \succ \varphi \to \phi \quad \Gamma \cup \{\psi\} \succ \chi}{\Gamma \cup \{(\varphi \to \phi) \to \psi\} \succ \chi}\ (\to\succ_4)$$

Table 1: The rules of LJ^*, for any formulae $\varphi, \psi, \chi, \phi$ and propositional variable P.

	$\Delta =$	$\mathcal{E}_P(\Delta)$ contains
(E0)	$\Delta' \cup \{\bot\}$	$\bot$
(E1)	$\Delta' \cup \{Q\}$	$E_P(\Delta') \wedge Q$
(E2)	$\Delta' \cup \{\phi \wedge \psi\}$	$E_P(\Delta' \cup \{\phi, \psi\})$
(E3)	$\Delta' \cup \{\phi \vee \psi\}$	$E_P(\Delta' \cup \{\phi\}) \vee E_P(\Delta' \cup \{\psi\})$
(E4)	$\Delta' \cup \{Q \to \phi\}$	$Q \to E_P(\Delta' \cup \{\phi\})$
(E5)	$\Delta' \cup \{P, P \to \phi\}$	$E_P(\Delta' \cup \{P, \phi\})$
(E6)	$\Delta' \cup \{((\phi \wedge \psi) \to \chi)\}$	$E_P(\Delta' \cup \{(\phi \to (\psi \to \chi))\})$
(E7)	$\Delta' \cup \{((\phi \vee \psi) \to \chi)\}$	$E_P(\Delta' \cup \{\phi \to \chi, \psi \to \chi\})$
(E8)	$\Delta' \cup \{((\phi \to \psi) \to \chi)\}$	$(E_P(\Delta' \cup \{\psi \to \chi\}) \to A_P(\Delta' \cup \{\psi \to \chi\}, \phi \to \psi))$ $\to E_P(\Delta' \cup \{\chi\})$

	$(\Delta, \varphi) =$	$\mathcal{A}_P(\Delta, \varphi)$ contains
(A1)	$(\Delta' \cup \{Q\}, \varphi)$	$A_P(\Delta', \varphi)$
(A2)	$(\Delta' \cup \{\phi \wedge \psi\}, \varphi)$	$A_P(\Delta' \cup \{\phi, \psi\}, \varphi)$
(A3)	$(\Delta' \cup \{\phi \vee \psi\}, \varphi)$	$(E_P(\Delta' \cup \{\phi\}) \to A_P(\Delta' \cup \{\phi\}, \varphi))$ $\wedge (E_P(\Delta' \cup \{\psi\}) \to A_P(\Delta' \cup \{\psi\}, \varphi))$
(A4)	$(\Delta' \cup \{Q \to \phi\}, \varphi)$	$Q \wedge A_P(\Delta' \cup \{\phi\}, \varphi)$
(A5)	$(\Delta' \cup \{P, P \to \phi\}, \varphi)$	$A_P(\Delta' \cup \{P, \phi\}, \varphi)$
(A6)	$(\Delta' \cup \{((\phi \wedge \psi) \to \chi)\}, \varphi)$	$A_P(\Delta' \cup \{(\phi \to (\psi \to \chi))\}, \varphi)$
(A7)	$(\Delta' \cup \{((\phi \vee \psi) \to \chi)\}, \varphi)$	$A_P(\Delta' \cup \{\phi \to \chi, \psi \to \chi\}, \varphi)$
(A8)	$(\Delta' \cup \{((\phi \to \psi) \to \chi)\}, \varphi)$	$(E_P(\Delta' \cup \{\psi \to \chi\}) \to A_P(\Delta' \cup \{\psi \to \chi\}, \phi \to \psi))$ $\wedge A_P(\Delta' \cup \{\chi\}, \varphi)$
(A9)	(Δ, Q)	Q
(A10)	$(\Delta' \cup \{P\}, P)$	$\top$
(A11)	$(\Delta, \phi \wedge \psi)$	$A_P(\Delta, \phi) \wedge A_P(\Delta, \psi)$
(A12)	$(\Delta, \phi \vee \psi)$	$A_P(\Delta, \phi) \vee A_P(\Delta, \psi)$
(A13)	$(\Delta, \phi \to \psi)$	$E_P(\Delta \cup \{\phi\}) \to A_P(\Delta \cup \{\phi\}, \psi)$

Table 2: The definitions of $\mathcal{E}_P(\Delta)$ and $\mathcal{A}_P(\Delta, \varphi)$, for formulae $\phi, \psi, \chi \neq \varphi$ and propositional variables $Q \neq P$

(iii) If $\vdash \Delta_1 \cup \Delta_2 \succ \varphi$ and $P \notin \mathsf{Var}(\Delta_1)$, then

(a) If $P \notin \mathsf{Var}(\varphi)$, then $\vdash \Delta_1 \cup \{E_P(\Delta_2)\} \succ \varphi$

(b) $\vdash \Delta_1 \cup \{E_P(\Delta_2)\} \succ A_P(\Delta_2, \varphi)$.

Having done so, we can define $\forall_P \varphi := A_P(\varnothing, \varphi)$ and Pitts' Theorem follows.

(i) can be easily seen from table 2 and proven via induction on $<_{\mathsf{wt}}$. (ii) is proven via simultaneous $<_{\mathsf{wt}}$-induction on $\Delta \cup \varphi$, by showing at each step, that for all $\psi \in \mathcal{E}_P(\Delta)$ and $\chi \in \mathcal{A}_P(\Delta, \varphi)$ we have $\vdash \Delta \succ \psi$ and $\vdash \Delta \cup \{\chi\} \succ \varphi$ for each case in table 2. Finally, (iii) is quite extensive and proven via induction on the rules of LJ^*. We look at three examplary cases:

(Atom): φ is a propositional variable and in $\Delta_1 \cup \Delta_2$. We have two cases:

$\varphi = P$: (a) holds trivially. For (b), since $P \notin \Delta_1$, we have $\Delta_2 = \Delta' \cup \{P\}$. Then from table 2 (A10) we have $\top \in \mathcal{A}_p(\Delta_2, P)$ and thus $A_p(\Delta_2, P) = \top$.

$\varphi \neq P$: If $\varphi \in \Delta_1$, (a) holds trivially via (Atom) and with $\Delta_1 = \Delta' \cup \{\varphi\}$ and (A9) and since (a) holds, (b) follows.
If $\varphi \in \Delta_2 =: \Delta' \cup \{\varphi\}$, then (a) follows by (A1) and as in the previous case, (b) follows with (A9) and (a).

($\perp \succ$): If $\perp \in \Delta_1$, both (a) and (b) follow immediately by ($\perp \succ$).
If $\perp \in \Delta_2$ both (a) and (b) follow immediately by (A0).

($\succ \wedge$): We have $\varphi = \varphi_1 \wedge \varphi_2$, $\vdash \Delta_1 \cup \Delta_2 \succ \varphi_i$ and by induction (a) and (b) hold for both $\vdash \Delta_1 \cup \Delta_2 \succ \varphi_i$.

(a) If $P \notin \mathsf{Var}(\varphi_i)$ then, since (a) holds for φ_i we have $\vdash \Delta_1 \cup \{E_p(\Delta_2)\} \succ \varphi_i$ and the claim follows by ($\succ \wedge$).

(b) Since (b) holds for the φ_i, we have

$$\vdash \Delta_1 \cup \{E_P(\Delta_2)\} \succ A_P(\Delta_2, \varphi_i)$$

and by ($\succ \wedge$) we have

$$\vdash \Delta_1 \cup \{E_P(\Delta_2)\} \succ A_P(\Delta_2, \varphi_1) \wedge A_P(\Delta_2, \varphi_2)$$

By (A11) we have

$$\vdash \{A_P(\Delta_2, \varphi_1) \wedge A_P(\Delta_2, \varphi_2)\} \succ A_P(\Delta_2, \varphi)$$

and thus (b) holds for φ.

The cases ($\wedge \succ$), ($\succ \vee_1$) and ($\succ \vee_2$) work similarly, the implicative cases however involve more subcases and are thus a lot more extensive. $\square$

As mentioned above, we define

$$\exists_P \varphi := \forall_Q(\forall_P(\varphi \to Q) \to Q).$$

Remark 4. As the previous proof is constructive, the functions $\exists_P$ and $\forall_P$ are in fact computable and referred to as *Pitts quantifiers.*

The existence of the existential Pitts quantifier already follows from – and is used extensively in – the proof for Pitts' Theorem. However, since the existence of the universal quantifier is sufficient, we will only take this one as "given" and proceed to show that defining the existential quantifier as above actually works as intended.

Corollary 1.2. *For each propositional variable P, intuitionistic propositional formula φ and propositional theory Γ with $P \notin \mathsf{Var}(\Gamma)$, we have*

(i) $\vdash_I \varphi\left[\frac{\psi}{P}\right] \to \exists_P \varphi$ *for any formula ψ,*

(ii) $\vdash_I \forall_P\left(\varphi\left[\frac{\psi}{Q}\right]\right)$ *iff* $\vdash_I (\forall_P\varphi)\left[\frac{\psi}{Q}\right]$*, provided $P \neq Q$ and $P, Q \notin \mathsf{Var}(\psi)$,*[11]

(iii) $\Gamma \vdash_I \exists_P \varphi \to \psi$ *iff* $\Gamma \vdash_I \varphi \to \psi$*, provided $P \notin \mathsf{Var}(\psi)$ and*

(iv) $\Gamma \vdash_I \psi \to \forall_P \varphi$ *iff* $\Gamma \vdash_I \psi \to \varphi$*.*

Proof.

(i) We have $\{\varphi(\psi), \forall_P(\varphi \to Q)\} \vdash_I \varphi \to Q\left[\frac{\psi}{P}\right]$ and thus $\{\varphi(\psi), \forall_P(\varphi \to Q)\} \vdash_I Q$. Therefore

$$\varphi(\psi) \vdash_I \forall_P(\varphi \to Q) \to Q$$

and since Q does not occur in φ (or ψ - remember that Q is supposed to be a generic new variable not occuring anywhere)

$$\varphi(\psi) \vdash_I \forall_Q(\forall_P(\varphi \to Q) \to Q)$$

which means $\vdash_I \varphi\left[\frac{\psi}{P}\right] \to \exists_P \varphi$.

(ii) We have $\forall_P \varphi \vdash_I \varphi$ and therefore $(\forall_P\varphi)\left[\frac{\psi}{Q}\right] \vdash_I \varphi\left[\frac{\psi}{Q}\right]$. Since $P \notin \mathsf{Var}\left((\forall_P\varphi)\left[\frac{\psi}{Q}\right]\right)$ we get $(\forall_P\varphi)\left[\frac{\psi}{Q}\right] \vdash_I \forall_P\left(\varphi\left[\frac{\psi}{Q}\right]\right)$.

For the converse, note that for any formulae ϕ, ϕ' and χ, we have

$$\phi \leftrightarrow \phi' \vdash_I \chi\left[\frac{\phi}{Q}\right] \leftrightarrow \chi\left[\frac{\phi'}{Q}\right].$$

By definition $\forall_P\left(\varphi\left[\frac{\psi}{Q}\right]\right) \vdash_I \varphi\left[\frac{\psi}{Q}\right]$ and thus

$$\left\{\psi \leftrightarrow Q, \forall_P\left(\varphi\left[\frac{\psi}{Q}\right]\right)\right\} \vdash_I \varphi\left[\frac{\psi}{Q}\right] \leftrightarrow \varphi\left[\frac{Q}{Q}\right]$$

[11] The proof for this is taken directly from [Pit92]

and hence

$$\left\{\psi \leftrightarrow Q, \forall_P \left(\varphi \left[\frac{\psi}{Q}\right]\right)\right\} \vdash_I \varphi.$$

Since P does not occur on the left side, we get

$$\left\{\psi \leftrightarrow Q, \forall_P \left(\varphi \left[\frac{\psi}{Q}\right]\right)\right\} \vdash_I \forall_P \varphi.$$

Now we can substitue Q by ψ throughout and get $\forall_P \left(\varphi \left[\frac{\psi}{Q}\right]\right) \vdash_I (\forall_P \varphi) \left[\frac{\psi}{Q}\right]$.

(iii) Let $\Gamma \vdash_I \varphi \to \psi$. We have

$$\Gamma \cup \{\exists_P \varphi\} \vdash_I \forall_Q (\forall_P (\varphi \to Q) \to Q),$$

therefore

$$\Gamma \cup \{\exists_P \varphi\} \vdash_I \forall_P (\varphi \to Q) \to Q \left[\frac{\psi}{Q}\right]$$

and by (ii), since $P \notin \mathsf{Var}(\psi)$,

$$\Gamma \cup \{\exists_P \varphi\} \vdash_I (\varphi \to \psi) \to \psi \left[\frac{P}{P}\right]$$

and thus $\Gamma \cup \{\exists_P \varphi\} \vdash_I \psi$, which means $\Gamma \vdash_I \exists_P \varphi \to \psi$.

For the converse, let $\Gamma \vdash_I \exists_P \varphi \to \psi$, then by (i) we have $\vdash_I \varphi \to \exists_P \varphi$ and thus $\Gamma \vdash_I \varphi \to \psi$.

(iv) The equivalency can be proven directly: We have
$\Gamma \vdash_I \psi \to \forall_P \varphi$ if and only if
$\Gamma \cup \{\psi\} \vdash_I \forall_P \varphi$ if and only if (by Pitts' Theorem)
$\Gamma \cup \{\psi\} \vdash_I \varphi \left[\frac{P}{P}\right]$ if and only if
$\Gamma \vdash_I \psi \to \varphi$. □

Interestingly, Pitts' Theorem allows us to easily adapt a well known proof for Craig's interpolation theorem for propositional logic to IpC. We will need interpolation later on to prove the amalgamation property for Heyting algebras.

Theorem 1.2 (Craig's interpolation theorem for IpC)**.** *If $\vdash_I \varphi \to \psi$, then there is an* interpolant*, meaning a formula χ such that*

$$\vdash_I \varphi \to \chi \text{ and } \vdash_I \chi \to \psi$$

with $\mathsf{Var}(\chi) \subseteq \mathsf{Var}(\varphi) \cap \mathsf{Var}(\psi)$.

Proof. Let $\vdash_I \varphi \to \psi$. We will proceed by induction on $|\mathsf{Var}(\varphi) \setminus \mathsf{Var}(\psi)| =: n$.
For $n = 0$, φ itself is suitable.
Let $n = m + 1$ and let the hypothesis hold for all $k \leq m$. Pick one $P \in \mathsf{Var}(\varphi) \setminus \mathsf{Var}(\psi)$. By corollary 1.2 we have $\vdash_I \varphi \to \exists_P \varphi$ and $\vdash_I \exists_P \varphi \to \psi$. Note that $\exists_P \varphi$ doesn't contain P anymore, which means $|\mathsf{Var}(\exists_P \varphi) \setminus \mathsf{Var}(\psi)| \leq m$. Hence, by the induction hypothesis there is an interpolant χ for $\exists_P \varphi \to \psi$ and since $\vdash_I \varphi \to \exists_P \varphi$ we can use χ as a suitable interpolant for $\varphi \to \psi$. □

2 Heyting algebras

Heyting algebras are to IpC as boolean algebras are to classical logic, a relationship that we will explore more detailed later on. In particular, Heyting algebras (and thus boolean algebras as well) are special kinds of lattices, so it makes sense to start with these.

2.1 Lattices and Heyting algebras

We can define lattices in two ways: As algebraic structures and via partial orderings. A *partial order* is a reflexive, antisymmetric and transitive relation.[12] All proofs in this section are taken from either [BS] or [BD74].

Definition 2.1. [13] A *lattice* is a structure in the language $L_L = (\sqcap, \sqcup)$ that satisfies the (universal closure of the)[14] following theory:

L1	$(x \sqcap y) \sqcap z \doteq x \sqcap (y \sqcap z)$	$(x \sqcup y) \sqcup z \doteq x \sqcup (y \sqcup z)$	Associativity
L2	$x \sqcap y \doteq y \sqcap x$	$x \sqcup y \doteq y \sqcup x$	Commutativity
L3	$x \sqcap x \doteq x$	$x \sqcup x \doteq x$	Idempotence
L4	$x \sqcap (x \sqcup y) \doteq x$	$x \sqcup (x \sqcap y) \doteq x$	Absorption

Remark 5. The axioms are *dual* in the sense that for each axiom, if we consistently replace $\sqcap$ by $\sqcup$ and vice versa, the resulting equation is again an axiom. This implies that any proof for a certain proposition yields a proof for the dual proposition by consistently taking for each statement in the proof the dual statement.

The second (equivalent) definition of lattices is the following:

Definition 2.2. [15] A *lattice* is a partially ordered set, in which for all elements a, b both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist.

Theorem 2.1. *Definition 2.1 and definition 2.2 coincide by defining the partial order $a \leq b$ iff $a = a \sqcap b$ or, respectively, the operations $a \sqcup b = \sup\{a, b\}$ and $a \sqcap b = \inf\{a, b\}$.*

Proof.

- Let $\mathfrak{L}$ be a lattice by definition 2.1 and $\leq$ be defined as above. By idempotence $a = a \sqcap a$, so $\leq$ is reflexive.
  If $a \leq b$ and $b \leq a$, we have $a = a \sqcap b = b \sqcap a = b$, thus $\leq$ is antisymmetric.
  If $a \leq b$ and $b \leq c$ we have $a = a \sqcap b = a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c = a \sqcap c$ and thus $a \leq c$. So $\leq$ is also transitiv and thus is a partial order.

[12][BS, p.6]
[13][BS, p.5]
[14]i.e. every variable is to be thought of as universally quantified
[15][BS, p.8]

We have by absorption $a = a \sqcap (a \sqcup b)$ and $b = b \sqcap (a \sqcup b)$, so $a, b \leq (a \sqcup b)$ and thus $a \sqcup b$ is an upper bound. Assume $a, b \leq u$, then $a \sqcup u = (a \sqcap u) \sqcup u = u$ and analogously $b \sqcup u = u$. We have $(a \sqcup u) \sqcup (b \sqcup u) = u \sqcup u$ and by associativity and idempotence $(a \sqcup b) \sqcup u = u$. Thus, $(a \sqcup b) \sqcap u = (a \sqcup b) \sqcap ((a \sqcup b) \sqcup u)$. By absorption, this is equal to $a \sqcup b$ and therefore $a \sqcup b \leq u$. Thus, $a \sqcup b = \sup\{a, b\}$. Analogously we can show, that $a \sqcap b = \inf\{a, b\}$.

- Let $\mathfrak{L}$ be a lattice by definition 2.2. Commutativity, associativity and idempotence follow directly from the definitions of sup and inf. Absorption follows easily by observing, that $a = \sup\{a, \inf\{a, b\}\} = \inf\{a, \sup\{a, b\}\}$. □

Definition 2.3. [16] A *distributive lattice* is a lattice that satisfies the axiom

D1 $\forall x \forall y \forall z\; x \sqcap (y \sqcup z) \doteq (x \sqcap y) \sqcup (x \sqcap z)$ or its dual

D2 $\forall x \forall y \forall z\; x \sqcup (y \sqcap z) \doteq (x \sqcup y) \sqcap (x \sqcup z)$.

One of both axioms is sufficient, since one implies the other:

Lemma 2.1. *D1 and D2 are equivalent.*

Proof. Assume D1 holds, then

$$\begin{aligned}
& x \sqcup (y \sqcap z) \\
=& (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) && \text{(Absorption)} \\
=& x \sqcup ((z \sqcap x) \sqcup (z \sqcap y)) && \text{(Associativity and commutativity)} \\
=& x \sqcup (z \sqcap (x \sqcup y)) && \text{(D1)} \\
=& (x \sqcap (x \sqcup y)) \sqcup (z \sqcap (x \sqcup y)) && \text{(Absorption)} \\
=& (x \sqcup y) \sqcap (x \sqcup z). && \text{(D1)}
\end{aligned}$$

By duality, (D2) also implies (D1). □

Next, we introduce a smallest and a largest element:

Definition 2.4. [17] A *bounded lattice* is a lattice with two distinguished elements 0 and 1 such that

B1 $\forall x\; x \sqcap 1 \doteq x$ and

B2 $\forall x\; x \sqcup 0 \doteq x$

hold.

Remark 6. In a bounded lattice, the duality principle again holds, if (in addition to $\sqcap$ and $\sqcup$) we exchange 0 and 1 (or to put it another way: the dual of a bounded lattice is again a bounded lattice). If existent, 0 and 1 are unique (since if $\mathfrak{L}$ is a bounded lattice, both $(\mathfrak{L}, \sqcap, 1)$ and $(\mathfrak{L}, \sqcup, 0)$ are monoids).

[16] [BS, p.12]
[17] [BD74, p.49]

Definition 2.5. [18] Let $\mathfrak{L}$ be a lattice and $x, y \in \mathfrak{L}$. If there is a largest element z, such that $x \sqcap z \leq y$, we call z the *relative pseudocomplement of x with respect to y*, denoted by $x \hookrightarrow y$.

This finally brings us to:

Definition 2.6. [19] A *Heyting algebra* is a bounded lattice, where for each two elements x, y the relative pseudocomplement $x \hookrightarrow y$ exists.

Example 2.1. [20] Examples of Heyting algebras are:

- Every boolean algebra is a Heyting algebra, by defining $x \hookrightarrow y = x^C \sqcup y$.
- Every chain with a least and largest element (0 and 1) is a Heyting algebra by defining $x \hookrightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$
- Let $\mathcal{T}$ be a topological space over some set X. With $U \hookrightarrow V := \mathsf{int}(U^C \cup V)$ (where int denotes the interior) and $\sqcap$ and $\sqcup$ as intersection and union respectively, $\mathcal{T}$ is a Heyting algebra.
- As we will show, the Tarski-Lindenbaum algebra of IpC is a Heyting algebra. Moreover, Heyting algebras are exactly the algebraic models of IpC-theories (in the sense of theorem 2.4).

Recall that $\sqcap$ and $\sqcup$ bind stronger than $\hookrightarrow$ (e.g. $x \sqcap y \hookrightarrow z$ is to be read as $(x \sqcap y) \hookrightarrow z$).

Lemma 2.2. [21] *In a Heyting algebra, the following statements hold for all x, y, z:*

(i) $x \sqcap (x \hookrightarrow y) \leq y$

(ii) $x \sqcap y \leq z \Leftrightarrow y \leq x \hookrightarrow z$

(iii) $x \leq y \Leftrightarrow x \hookrightarrow y = 1$

(iv) $y \leq x \hookrightarrow y$

(v) $x \leq y \Rightarrow z \hookrightarrow x \leq z \hookrightarrow y$ *and* $y \hookrightarrow z \leq x \hookrightarrow z$

(vi) $x \hookrightarrow (y \hookrightarrow z) = x \sqcap y \hookrightarrow z$

(vii) $x \sqcap (y \hookrightarrow z) = x \sqcap (x \sqcap y \hookrightarrow x \sqcap z)$

(viii) $x \sqcap (x \hookrightarrow y) = x \sqcap y$

(ix) $(x \sqcup y) \to z = (x \hookrightarrow z) \sqcap (y \hookrightarrow z)$

[18] [BD74, p.173]
[19] [BD74, p.174]
[20] [BD74, p.177]
[21] [BD74, p.174]

(x) $x \hookrightarrow y \sqcap z = (x \hookrightarrow y) \sqcap (x \hookrightarrow z)$

(xi) $(x \hookrightarrow y) \hookrightarrow 0 = ((x \hookrightarrow 0) \hookrightarrow 0) \wedge (y \hookrightarrow 0)$

Proof.

(i) Holds by definition.

(ii) $\Rightarrow$: holds by definition ($x \hookrightarrow z$ is the largest element with this property).
$\Leftarrow$: If $y \leq x \hookrightarrow z$, then $x \sqcap y \leq x \sqcap x \hookrightarrow z \leq z$.

(iii) We have $x \hookrightarrow y = 1 \Leftrightarrow 1 \leq x \hookrightarrow y \Leftrightarrow x \sqcap 1 \leq x \sqcap (x \hookrightarrow y) \leq y$.

(iv) $x \sqcap y \leq y$, therefore by (ii) $y \leq x \hookrightarrow y$.

(v) If $x \leq y$, then $z \sqcap (z \hookrightarrow x) \leq x \leq y$, so by (ii) $z \hookrightarrow x \leq z \hookrightarrow y$. Also, we have $x \sqcap (y \hookrightarrow z) \leq y \sqcap (y \hookrightarrow z) \leq z$, so again by (ii) we get $y \hookrightarrow z \leq x \hookrightarrow z$.

(vi) We have

$$x \sqcap y \sqcap (x \hookrightarrow (y \hookrightarrow z)) = y \sqcap (x \sqcap (x \hookrightarrow (y \hookrightarrow z))) \leq y \sqcap (y \hookrightarrow z) \leq z,$$

so by (ii) $x \hookrightarrow (y \hookrightarrow z) \leq x \sqcap y \hookrightarrow z$.
Conversely, $y \sqcap x \sqcap (x \sqcap y \hookrightarrow z) \leq z$, so by (ii) $x \sqcap (x \sqcap y \rightarrow z) \leq y \hookrightarrow z$ and therefore again by (ii) $x \sqcap y \hookrightarrow z \leq x \hookrightarrow (y \hookrightarrow z)$.

(vii) We have $(x \sqcap y) \sqcap x \sqcap (y \hookrightarrow z) \leq x \sqcap z$, so with (ii) we get $x \sqcap (y \hookrightarrow z) \leq x \sqcap y \hookrightarrow x \sqcap z$ and thus

$$x \sqcap x \sqcap (y \hookrightarrow z) \leq x \sqcap (x \sqcap y \hookrightarrow y \sqcap z).$$

Conversely, $x \sqcap (x \sqcap y \hookrightarrow x \sqcap z) \leq x$ and $y \sqcap x \sqcap (x \sqcap y \hookrightarrow x \sqcap z) \leq x \sqcap z \leq z$, so by (ii) $x \sqcap (x \sqcap y \hookrightarrow x \sqcap z) \leq y \hookrightarrow z$ and therefore $x \sqcap x \sqcap (x \sqcap y \hookrightarrow x \sqcap z) \leq x \sqcap (y \hookrightarrow z)$.

(viii) By definition $x \sqcap (x \hookrightarrow y) \leq x, y$ and $x \sqcap y \leq x, x \hookrightarrow y$ (the latter by (ii)).

(ix) We have $x, y \leq x \sqcup y$ and thus by (v) $x \sqcup y \hookrightarrow z \leq x \hookrightarrow z, y \hookrightarrow z$.
Conversely,

$$\begin{aligned}
&(x \sqcup y) \sqcap (x \hookrightarrow z) \sqcap (y \hookrightarrow z) \\
\leq &(x \sqcap (x \hookrightarrow z) \sqcap (y \hookrightarrow z)) \sqcup (y \sqcap (x \hookrightarrow z) \sqcap (y \hookrightarrow z)) \\
\leq &(x \sqcap (x \hookrightarrow z)) \sqcup (y \sqcap (y \hookrightarrow z)) \\
\leq &z \sqcup z = z,
\end{aligned}$$

so by (ii) $(x \hookrightarrow z) \sqcap (y \hookrightarrow z) \leq (x \sqcup y) \hookrightarrow z$.

(x) We have $y \sqcap z \leq y, z$ and thus by (v) $x \hookrightarrow y \wedge z \leq x \hookrightarrow y, x \hookrightarrow z$. Conversely, we have

$$x \sqcap (x \hookrightarrow y) \sqcap (x \hookrightarrow z) \leq x \sqcap y \sqcap (x \hookrightarrow z) \leq y \sqcap z,$$

so by (ii) we get $(x \hookrightarrow y) \sqcap (x \hookrightarrow z) \leq x \hookrightarrow (y \sqcap z)$.

(xi) By (iv) $y \leq x \hookrightarrow y$ and by (v) $(x \hookrightarrow y) \hookrightarrow 0 \leq y \hookrightarrow 0$. Since $0 \leq y$ we get by applying (v) twice $x \hookrightarrow 0 \leq x \hookrightarrow y$ and $(x \hookrightarrow y) \hookrightarrow 0 \leq (x \hookrightarrow 0) \hookrightarrow 0$. Combining both, we get $(x \hookrightarrow y) \hookrightarrow 0 \leq ((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0)$.
Conversely,

$$\begin{aligned} &((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0) \sqcap (x \rightarrow y) \\ \leq &((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0) \sqcap ((y \hookrightarrow 0) \sqcap x \rightarrow (y \hookrightarrow 0) \sqcap y) \text{ (by (vii))} \end{aligned}$$

which is equal to

$$\begin{aligned} &((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0) \sqcap ((y \hookrightarrow 0) \sqcap x \rightarrow 0) \\ = &((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0) \sqcap ((y \hookrightarrow 0) \sqcap x \rightarrow (y \hookrightarrow 0) \sqcap 0) \\ = &((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0) \sqcap (x \hookrightarrow 0) \text{ (again by (vii))} \end{aligned}$$

which is equal to 0. So by (ii) we have $((x \hookrightarrow 0) \hookrightarrow 0) \sqcap (y \hookrightarrow 0) \leq (x \hookrightarrow y) \hookrightarrow 0$. □

With these, we can now give an equational axiomatization for Heyting algebras:

Theorem 2.2. *Heyting algebras are exactly the models of the (universal closure of the) following equational theory T_H in the language $L_H = (1, 0, \sqcap, \sqcup, \hookrightarrow)$:*
The axioms for a bounded distributive lattice (i.e. L1–L4, D1 (or D2), B1 and B2) and

H1 $x \sqcap (x \hookrightarrow y) \doteq x \sqcap y$

H2 $x \sqcap (y \hookrightarrow z) \doteq x \sqcap (x \sqcap y \hookrightarrow x \sqcap z)$

H3 $x \sqcap y \hookrightarrow x \doteq 1$

Proof. We have to show the following:

- Every Heyting algebra satisfies H1–H3:
  H1 is lemma 2.2.(viii), H2 is lemma 2.2.(vii). and H3 follows from lemma 2.2.(iii).
- In every bounded distributive lattice satisfying H1–H3, $x \hookrightarrow y$ is the relative pseudocomplement:
  We have by H1 $x \sqcap (x \hookrightarrow y) = x \sqcap y \leq y$. Suppose $x \sqcap z \leq y$ for some z (and thus $x \sqcap y \sqcap z = x \sqcap z$), then by H2 and H3

  $$z \sqcap (x \hookrightarrow y) = z \sqcap (z \sqcap x \hookrightarrow z \sqcap y) = z \sqcap (x \sqcap y \sqcap z \hookrightarrow y \sqcap z) = z$$

  and therefore $z \leq (x \hookrightarrow y)$. □

2.2 Heyting algebras as models of IpC

We start by looking at the *freely generated* Heyting algebras. Given a set of elements ("generators") A, the set of $L_H(A)$-terms $\mathcal{T}(A)$ obviously coincides with the set of propositional formulae $\mathcal{F}(A)$ (with the elements of A as propositional variables) via the following recursively defined bijection:

$$[\cdot] : \mathcal{F}(A) \to \mathcal{T}(A)$$
$$[P] = P \text{ for } P \in A$$
$$[\varphi \wedge \psi] = [\varphi] \sqcap [\psi]$$
$$[\varphi \vee \psi] = [\varphi] \sqcup [\psi]$$
$$[\varphi \to \psi] = [\varphi] \hookrightarrow [\psi]$$
$$[\top] = 1 \qquad [\bot] = 0$$

We now define an equivalence relation on $\mathcal{T}(A)$, by letting for any terms t_1 and t_2:

$$t_1 \sim t_2 \text{ iff } T_H \models t_1 \doteq t_2.$$

Since T_H is equational, the quotient algebra $\mathcal{T}(A)/\sim$ is a Heyting algebra[22], the *freely generated Heyting algebra over A*, denoted by $\mathcal{H}_A$.

Theorem 2.3. *The freely generated Heyting algebra $\mathcal{H}_A$ is the* Tarski-Lindenbaum algebra *of IpC over A. This means for any formulae $a, b \in \mathcal{F}(A)$ we have $[a] \sim [b]$ iff $\vdash_I a \leftrightarrow b$.*

Proof. We have to show the following:

1. For every axiom $[t_1] \doteq [t_2]$ of T_H we have $\vdash_I t_1 \leftrightarrow t_2$.
2. *Modus Ponens:* If $[\varphi] = 1$ and $[\varphi] \hookrightarrow [\psi] = 1$, then $[\psi] = 1$.
3. For every axiom $\varphi(\psi_1, \psi_2, \psi_3)$ of IpC we have $\mathcal{H}_A \models \forall x, y, z[\varphi(x, y, z)] \doteq 1$.

Proof of 1. We have already proven, that in IpC conjunction is commutative and associative, the same holds for disjunction (follows immediately by IpC5, IpC6 and Modus Ponens). The axioms for 1 and 0 hold by definition of $\top$ and $\bot$.

Idempotence: By IpC4 and IpC5 we have $\vdash_I x \to (x \vee x)$ and $\vdash_I (x \wedge x) \to x$. IpC8 gives us

$$\vdash_I (x \to x) \to ((x \to x) \to ((x \vee x) \to x))$$

and therefore $\vdash_I (x \vee x) \to x$. By IpC3 we have $\vdash_I x \to (x \to (x \wedge x))$ and thus $\vdash_I x \to (x \wedge x)$.

Absorption: We have $\vdash_I x \wedge (x \vee y) \to x$ (IpC4), $\vdash_I x \to x \vee y$ (IpC5), $\vdash_I x \to x$ and thus $\vdash_I x \to (x \wedge (x \vee y))$. The converse is IpC4.

Distributivity: We only need to show one distributive law: We have by IpC8

$$\vdash_I (y \to ((x\wedge y)\vee(x\wedge z))) \to ((z \to ((x\wedge y)\vee(x\wedge z))) \to ((y\vee z) \to ((x\wedge y)\vee(x\wedge z))))$$

and $x \wedge (y \vee z) \vdash_I x$ and thus $x \wedge (y \vee z) \vdash_I y \to (x \wedge y)$. Therefore

$$x \wedge (y \vee z) \vdash_I y \to ((x \wedge y) \vee (x \wedge z))$$

[22] For a more detailled exploration of equational theories and (fully invariant) congruence relations, see [BS, p.99ff]

(analogously for z instead of y). Thus $x \wedge (y \vee z) \vdash_I ((x \wedge y) \vee (x \wedge z))$. Conversely, we have

$$\vdash_I ((x \wedge y) \rightarrow x) \rightarrow (((x \wedge z) \rightarrow x) \rightarrow (((x \wedge y) \vee (x \wedge z)) \rightarrow x))$$

and hence $(x\wedge y)\vee(x\wedge z) \vdash_I x$. Analogously we can show $(x\wedge y)\vee(x\wedge z) \vdash_I y\vee z$ and the claim follows.

H1: We have $(x \wedge (x \rightarrow y)) \vdash_I x$ and $x \wedge (x \rightarrow y) \vdash_I x \rightarrow y$ and by Modus Ponens and IpC3 $x \wedge (x \rightarrow y) \vdash_I x \wedge y$. The converse follows directly by IpC1.

H2: By Modus Ponens $\{x \wedge (y \rightarrow z), x \wedge y\} \vdash_I z$ and thus

$$x \wedge (y \rightarrow z) \vdash_I x \wedge y \rightarrow x \wedge z.$$

For the converse,

$$\{x \wedge (x \wedge y \rightarrow x \wedge z), y\} \vdash_I x \wedge y$$

and by Modus Ponens and IpC3

$$\{x \wedge (x \wedge y \rightarrow x \wedge z), y\} \vdash_I z.$$

Hence, $x \wedge (x \wedge y \rightarrow x \wedge z) \vdash_I y \rightarrow z$.

H3: Follows from the fact, that $x \wedge y \rightarrow x$ is an axiom (IpC4).

Proof of 2. Let $[\varphi] = [\varphi] \hookrightarrow [\psi] = 1$.
We then have $[\varphi] \sqcap ([\varphi] \hookrightarrow [\psi]) \leq [\psi]$, ergo $1 \sqcap 1 \leq [\psi]$ and thus $\mathcal{H}_A \models 1 \doteq [\psi]$.

Proof of 3. IpC1: $\varphi \rightarrow (\psi \rightarrow \varphi)$
We have by lemma 2.2.(iv) $x \leq y \hookrightarrow x$ and thus by lemma 2.2.(iii) $x \hookrightarrow (y \hookrightarrow x) = 1$.

IpC2: $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi))$
We have by lemma 2.2.(vi) $x \hookrightarrow (y \hookrightarrow z) = (x \sqcap y) \hookrightarrow z$ and

$$(x \hookrightarrow y) \hookrightarrow (((x\sqcap y) \hookrightarrow z) \hookrightarrow (x \hookrightarrow z)) = ((x \hookrightarrow y)\sqcap((x\sqcap y) \hookrightarrow z)) \hookrightarrow (x \hookrightarrow z)$$

and thus $((x \hookrightarrow y) \sqcap ((x \sqcap y) \hookrightarrow z)) \leq (x \hookrightarrow z)$ which holds iff

$$x \hookrightarrow y \leq ((x \sqcap y) \hookrightarrow z) \hookrightarrow (x \hookrightarrow z)$$

and therefore $(x \hookrightarrow y) \hookrightarrow (((x \sqcap y) \hookrightarrow z) \hookrightarrow (x \hookrightarrow z)) = 1$.

IpC3: $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
We have $(x \hookrightarrow (y \hookrightarrow (x \sqcap y)) = x \sqcap y \hookrightarrow x \sqcap y = 1$, since $x \sqcap y \leq x \sqcap y$.

IpC4: $(\varphi \wedge \psi) \rightarrow \varphi$
Holds, since $x \sqcap y \leq x$.

IpC5: $\varphi \rightarrow (\varphi \vee \psi)$
Holds, since $x \leq x \sqcup y$.

IpC6: $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
We have by lemma 2.2.(vi)

$$(x \hookrightarrow z) \hookrightarrow ((y \hookrightarrow z) \hookrightarrow ((x \sqcup y) \hookrightarrow z)) = ((x \hookrightarrow z) \sqcap (y \hookrightarrow z)) \hookrightarrow ((x \sqcup y) \hookrightarrow z))$$

and by lemma 2.2.(ix) $(x \sqcup y) \hookrightarrow z = (x \hookrightarrow z) \sqcap (y \hookrightarrow z)$.

IpC7: $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$
We need to show

$$\begin{aligned} 1 =& (x \hookrightarrow y) \hookrightarrow ((x \hookrightarrow (y \hookrightarrow 0)) \hookrightarrow (x \hookrightarrow 0)) \\ =& ((x \hookrightarrow y) \sqcap (x \hookrightarrow (y \hookrightarrow 0))) \hookrightarrow (x \hookrightarrow 0) \end{aligned}$$

(lemma 2.2.(vi)) which holds iff

$$\begin{aligned} & (x \hookrightarrow y) \sqcap (x \hookrightarrow (y \hookrightarrow 0)) \sqcap (x \hookrightarrow 0) = (x \hookrightarrow y) \sqcap (x \hookrightarrow (y \hookrightarrow 0)) \\ \Leftrightarrow \quad & x \hookrightarrow (y \sqcap (y \hookrightarrow 0) \sqcap 0) = x \hookrightarrow (y \sqcap (y \hookrightarrow 0)) \quad \text{(lemma 2.2.(x))} \\ \Leftrightarrow \quad & x \hookrightarrow (y \sqcap 0) = x \hookrightarrow (y \sqcap 0) \end{aligned}$$

IpC8: $\varphi \to (\neg\varphi \to \psi)$
We have

$$x \hookrightarrow ((x \hookrightarrow 0) \hookrightarrow y) = (x \sqcap (x \hookrightarrow 0)) \hookrightarrow y = 0 \hookrightarrow y$$

(lemma 2.2.(vi)) which holds since $0 \leq y$. □

All other (i.e. not freely generated) Heyting algebras are covered by the following theorem:

Theorem 2.4. *Given any IpC-theory Γ over a set of propositional variables A, the set of all propositional formulae $\mathcal{F}(A)/\Gamma$ over A divided by the equivalence relation*

$$\varphi \sim_\Gamma \psi \text{ iff } \Gamma \vdash_I \varphi \leftrightarrow \psi$$

is a Heyting algebra. Conversely, we can interpret any Heyting algebra $\mathfrak{H}$ as $\mathcal{F}(H)/H^{=1}$, where $H^{=1} := \{[t] \mid t \in \mathcal{F}(H) \text{ and } \mathfrak{H} \models [t] \doteq 1\}$.

Proof. That $\mathcal{F}(A)/\Gamma$ is a Heyting algebra can be easily seen by the fact that $\vdash_I \varphi \Rightarrow \Gamma \vdash_I \varphi$, and thus $\sim \subseteq \sim_\Gamma$ (where $\sim$ is the equivalence relation as defined above) - since we only divide by additional equations, the equations from T_H still hold if we divide by $\sim_\Gamma$.

For a more rigorous proof, we can take the freely generated Heyting algebra $\mathcal{H}_A$ and use the well known fact, that the filters in a Heyting algebra $\mathfrak{H}$ are isomorphic to the congruence relations on $\mathfrak{H}$.[23] We can w.l.o.g. assume Γ to be deductively closed (i.e. if $\Gamma \vdash_I \varphi$, then $\varphi \in \Gamma$). We then have:

1. If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$, then $\psi \in \Gamma$ and

[23] See [BD74, p.178f]

2. If $\varphi \in \Gamma$ and $\psi \in \Gamma$, then $\varphi \wedge \psi \in \Gamma$.

This shows, that the image of Γ in $\mathcal{H}_A$ is a filter on $\mathcal{H}_A$ and thus, that the equivalence relation generated by this filter is a congruence relation.

It remains to show, that $\mathfrak{H} \models [t] \doteq 1$ iff $H^{=1} \vdash_I t$. The implication from left to right holds by definition, the other direction holds, as (as we have shown) T_H entails all IpC-axioms and Modus Ponens. □

Given this translation between IpC-theories and Heyting algebras, we can define:

Definition 2.7. The *polynomial Heyting algebra* $\mathfrak{H}(X)$ over a Heyting algebra $\mathfrak{H}$ is the Heyting algebra $\mathcal{F}(H \cup \{X\})/H^{=1}$ for some $X \notin H$.

3 Model companions and completions

A *model completion* for a certain theory (of first order *predicative* formulae) is a special kind of *model companion*, which is a related *model complete* theory.

Definition 3.1. [24] A theory T is *model complete* if for every model $\mathfrak{A} \models T$, every substructure $\mathfrak{B}$ which is a model of T is an elementary substructure (i.e. $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same $L(B)$-sentences).

Any theory with quantifier elimination is model complete.[25] The classic example for a model complete theory (and the following concepts) is the theory of algebraically closed fields ACF. Since ACF does not determine the characteristic of a model, it is not complete. However, given a certain (algebraically closed) field, any algebraically closed extension or substructure has the same characteristic; all other L_{Field}-sentences are already decided by the theory. As such, ACF is the model completion of the theory of fields:

Definition 3.2.

- [26] A theory T^* is a *model companion* of a theory T if the following conditions are satisfied:

  (a) Each model of T can be extended to a model of T^* and vice versa,

  (b) T^* is model complete.

- [27] A *model completion* T^* of a theory T is a model companion of T with the following additional property:
  For all models $\mathfrak{A} \models T$ and $\mathfrak{A}_1, \mathfrak{A}_2 \models T^*$:

$$\text{If } \mathfrak{A} \subseteq \mathfrak{A}_1, \mathfrak{A}_2, \text{ then } (\mathfrak{A}_1, A) \equiv (\mathfrak{A}_2, A).$$

If a model companion exists, it is unique (up to equivalence, of course). Consequently, we are only interested in whether one exists or not.

Theorem 3.1. *Any theory has at most one model companion.*

Proof. [28] Assume a theory T has two model companions T_1 and T_2 and let $\mathfrak{A}_0 \models T_1$, then $\mathfrak{A}_0$ can be embedded in a model $\mathfrak{B}_0 \models T_2$, which can in turn be embedded in a model $\mathfrak{A}_1 \models T_1$ and so on, resulting in two elementary chains $(\mathfrak{A}_i)_{i\in\omega}$ and $(\mathfrak{B}_i)_{i\in\omega}$. Since $\mathfrak{A}_i \prec \mathfrak{B}_i$ and $\mathfrak{B}_i \prec \mathfrak{A}_{i+1}$, we have $\bigcup_{i\in\omega} \mathfrak{A}_i = \bigcup_{i\in\omega} \mathfrak{B}_i = \mathfrak{C}$. Since $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are elementary substructures of $\mathfrak{C}$, we have $\mathfrak{A}_0 \equiv \mathfrak{B}_0$ and thus $\mathfrak{A}_0 \models T_2$. Analogously we can show that every model of T_2 is a model of T_1. □

[24] [TZ12, p.34]
[25] [TZ12, p.34]
[26] [TZ12, p.35]
[27] [Pot81, p.106]
[28] [TZ12, p.35]

Other examples for model companions or completions are:

Example 3.1. [29]

- The theory of differentially closed fields is the model companion of the theory of differential fields.
- The theory of the random graph is the model completion of the theory of graphs.
- [30] The theory of atomless boolean algebras is the model completion of the theory of boolean algebras.

If a model companion of T exists, its models are exactly the T-existentially closed structures, as in the following definition:

Definition 3.3. [31]

- A substructure $\mathfrak{A} \subseteq \mathfrak{B}$ is called *existentially closed* in $\mathfrak{B}$ if for every existential $L(A)$-sentence φ,
$$\mathfrak{B} \models \varphi \Rightarrow \mathfrak{A} \models \varphi.$$
- A structure $\mathfrak{A}$ is called *T-existentially-closed* (T-e.c.) if $\mathfrak{A}$ can be embedded in a model of T and is existentially closed in every extension which is a model of T.

The previous definition results in the following useful criterion for the existence of a model companion:

Theorem 3.2. *T has a model companion iff the class of T-existentially-closed structures is an elementary (i.e. axiomatizable) class.*

Proof. The proof uses quite a lot of model theory. The details can be found in [TZ12, p.35ff].

Assume T has a model companion T^*. Since T^* is model complete, it is in particular inductive, and thus axiomatizable by $\forall\exists$-formulae. This implies that T^* is contained in the *Kaiser hull* of T, which is the biggest inductive theory T^{KH} with $T_\forall = T_\forall^{KH}$ (where $T_\forall$ is the universal part of T). The Kaiser hull happens to be exactly the $\forall\exists$-part of the theory of all T-e.c. structures.

So, let $\mathfrak{M} \models T^*$ and $\mathfrak{A} \models T$ an extension of $\mathfrak{M}$. $\mathfrak{A}$ can be embedded in a model $\mathfrak{N} \models T^*$ and since $\mathfrak{M} \prec \mathfrak{N}$, $\mathfrak{M}$ is existentially closed in $\mathfrak{A}$. This shows that all models of T^{KH} are T-e.c., and since all T-e.c. structures are models of T^{KH}, the Kaiser hull serves as an axiomatization of the class of T-e.c. structures.

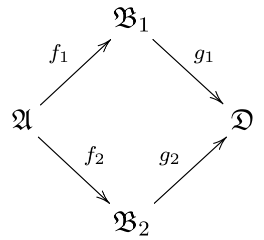
For the converse, let T^+ be an axiomatization of the class of T-e.c. structures. Robinson's test then tells us that T^+ is model complete (since all of its models are by definition existentially closed) and thus serves as the model companion. $\square$

[29] [TZ12, p.37ff]
[30] [CK90, 197]
[31] [TZ12, p.35]

To show that a model companion is in deed a model completion, we can use the following property:

Definition 3.4. [32]

- A class $\mathcal{K}$ of structures has the *amalgamation property* if for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ and embeddings $f_i : A \to B_i, (i = 1, 2)$, there is some $\mathfrak{D} \in \mathcal{K}$ and two embeddings $g_i : B_i \to \mathfrak{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$, i.e. the following diagram commutes.



- A theory T has the amalgamation property if the class $\mathsf{Mod}(T)$ of all models of T has the amalgamation property.

Remark 7. Since the f_i in the above definition are embeddings, we can always w.l.o.g. assume them to be the identity, i.e. we can assume $\mathfrak{A}$ is a common substructure of $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

As can be easily seen, we have:

Theorem 3.3. *Let T^* be the model companion of T. Then T^* is a model completion of T iff T has the amalgamation property.*

Proof. Assume T has the amalgamation property and let $\mathfrak{A} \models T$ be a common substructure of $\mathfrak{A}_1, \mathfrak{A}_2 \models T^*$. We can w.l.o.g. assume $\mathfrak{A}_1, \mathfrak{A}_2 \models T$ (since they can be extended to models of T). So there is a model $\mathfrak{D} \models T$ with $\mathfrak{A}_1, \mathfrak{A}_2 \subseteq \mathfrak{D}$. We can again w.l.o.g. assume $\mathfrak{D} \models T^*$ and since T^* is model complete, we have $\mathfrak{A}_1, \mathfrak{A}_2 \prec \mathfrak{D}$ and thus $(\mathfrak{A}_1, A) \equiv (\mathfrak{A}_2, A)$.

For the converse, assume T^* is a model completion and let $\mathfrak{A}$ be w.l.o.g. the largest common substructure of $\mathfrak{A}_1, \mathfrak{A}_2 \models T$. We can w.l.o.g. assume $\mathfrak{A}_1, \mathfrak{A}_2 \models T^*$. Then $(\mathfrak{A}_1, A) \equiv (\mathfrak{A}_2, A)$, which means $\mathsf{Th}(\mathfrak{A}, A) = \mathsf{Th}(\mathfrak{A}_1, A_1) \cap \mathsf{Th}(\mathfrak{A}_2, A_2)$. The claim then follows immediately by *joint consistency*; a corollary to Craig's interpolation theorem (in this case for the predicate calculus), which states that for any complete (and consistent) theory T with two complete (and consistent) extensions T_1 and T_2, the union of T_1 and T_2 is again consistent:

Assume there were some φ such that $T_1 \cup T_2 \models \varphi \wedge \neg\varphi$. Then (by the compactness theorem) there are finite subset $\Gamma_1 \subset T_1$ and $\Gamma_2 \subset T_2$ with

$$\models \bigwedge_{\psi \in \Gamma_1} \psi \to \left(\bigwedge_{\psi \in \Gamma_2} \psi \to \varphi \wedge \neg\varphi \right).$$

[32] [TZ12, p.56]

By the interpolation theorem there is an interpolant χ in the language of T, which – since T_1 is an extension of T and the theories are complete – has to hold in T_1 and T and thus also in T_2. This implies $T_2 \models \varphi \wedge \neg\varphi$, ergo T_2 is inconsistent, contradiction. □

4 The model completion of T_H

In order to show that T_H has a model completion, we need to show, that the class of existentially closed Heyting algebras is an elementary class and that T_H has the amalgamation property. We start with the latter:

Theorem 4.1. *T_H has the amalgamation property.*

Proof. Let $\mathfrak{A} \models T_H$ be a common substructure of two Heyting algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$, and let $f_i : \mathfrak{A} \to \mathfrak{B}_i$ the corresponding embeddings. We may w.l.o.g. assume $\mathfrak{A}$ to be the largest common substructure of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. We can interpret $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as $\mathcal{F}(B_1)/B_1^{=1}$ and $\mathcal{F}(B_2)/B_2^{=1}$ respectively, and have $\mathfrak{A} \prec \mathfrak{B}_1, \mathfrak{B}_2$ and $A^{=1} \subseteq B_1^{=1}, B_2^{=1}$. Now consider $\mathcal{F}(B_1 \cup B_2)/(B_1^{=1} \cup B_2^{=1}) =: \mathfrak{D}$. We have to show, that the resulting canonic maps $g_i : \mathfrak{B}_i \to \mathfrak{D}$ are injective.

So, assume (w.l.o.g.) $g_1([b_1]) = g_1([b_2])$ for some $[b_1], [b_2] \in B_1$. Consequently,

$$(B_1^{=1} \cup B_2^{=1}) \vdash_I b_1 \leftrightarrow b_2$$

and by corollary 1.1 there are finite subsets $\Gamma_1 \subseteq B_1^{=1}$ and $\Gamma_2 \subseteq B_2^{=1}$ such that

$$\vdash_I \underbrace{\bigwedge_{\varphi\in\Gamma_2} \varphi}_{\in\mathcal{F}(B_2)} \to \underbrace{\left(\bigwedge_{\varphi\in\Gamma_1} \varphi \to (b_1 \leftrightarrow b_2)\right)}_{\in\mathcal{F}(B_1)}.$$

By theorem 1.2 (interpolation) we get a formula $\psi \in \mathcal{F}(B_1 \cap B_2) = \mathcal{F}(A)$ such that

$$\vdash_I \bigwedge_{\varphi\in\Gamma_2} \varphi \to \psi \text{ and } \vdash_I \psi \to \left(\bigwedge_{\varphi\in\Gamma_1} \varphi \to (b_1 \leftrightarrow b_2)\right)$$

Then $\mathfrak{B}_2 \models \left[\bigwedge_{\varphi\in\Gamma_2} \varphi\right] \doteq 1$ and thus $\mathfrak{B}_2 \models f_2([\psi]) \doteq 1$, which means (since f_2 is an embedding) $\mathfrak{A} \models [\psi] \doteq 1$ and hence $\mathfrak{B}_1 \models [b_1] \doteq [b_2]$.

Therefore, the g_i are injective and hence embeddings. □

To axiomatize the class of e.c. Heyting algebras, we will rely heavily on Pitts' theorem, so first, we will look at its consequences for T_H.

Remark 8. In the rest of this section, we deliberately do *not* differentiate between propositional formulae and L_H-terms, since doing so would lead to obfuscation rather than clarity. From the context it should always be clear, which of both is the intended meaning.

Recall that the Pitts quantifiers are computable, so extending L_H (and accordingly T_H) by the binary function symbols $\forall_x$ and $\exists_x$ is an extension by definition and hence conservative and unproblematic.

As mentioned in the introduction, our proof for the existence of the model companion is outlined in [GZ97].

Theorem 4.2. *For a propositional formula $t(\overline{a}, x)$ with propositional variables $\overline{a}$ from a Heyting algebra $\mathfrak{H}$ we have*

(i) $\mathfrak{H} \models \exists_x(t) \doteq 1$ *iff* $\mathfrak{H}(X)/t(X)$ *is an extension of* $\mathfrak{H}$, *where* $\mathfrak{H}(X)/t(X)$ *is the polynomial Heyting algebra* $\mathfrak{H}(X)$ *divided by the congruence generated by the equation* $t \doteq 1\left[\frac{X}{x}\right]$,

(ii) $\mathfrak{H} \models \forall_x(t) \doteq 1$ *iff the equation* $t \doteq 1\left[\frac{X}{x}\right]$ *holds in* $\mathfrak{H}(X)$.

Proof. We can interpret $\mathfrak{H}$ as $\mathcal{F}(H)/H^{=1}$ and w.l.o.g. assume $\mathfrak{H} \not\models 0 \doteq 1$.

(i) Let $\mathfrak{H} \models \exists_x(t) \doteq 1$, $\pi : \mathfrak{H} \to \mathfrak{H}(X)/t(X)$ the canonic map and $a, b \in H$ with $\pi(a) = \pi(b)$, then

$$(H^{=1} \cup \{t(X)\}) \vdash_I a \leftrightarrow b$$

and thus

$$H^{=1} \vdash_I t(X) \to (a \leftrightarrow b).$$

By corollary 1.2 we get

$$H^{=1} \vdash_I \exists_x t \to (a \leftrightarrow b)$$

and since $\exists_x(t) \doteq 1$ holds in $\mathfrak{H}$ we have $\mathfrak{H} \models a \doteq b$. Thus π is injective and hence $\mathfrak{H}(X)/t(X)$ an extension.

For the converse, let $\mathfrak{H}(X)/t(X)$ be an extension of $\mathfrak{H}$ and π the corresponding embedding, then with corollary 1.2

$$(H^{=1} \cup \{t(X)\}) \vdash_I \exists_x t$$

and therefore

$$\mathfrak{H}(X)/t(X) \models \exists_x(t) \doteq 1.$$

Hence, we can conclude that $\pi(\exists_x(t)) = \pi(1^{\mathfrak{H}})$ and since π is an embedding we have $\mathfrak{H} \models \exists_x(t) \doteq 1$.

(ii) The equivalency can be shown directly: We have $\mathfrak{H} \models \forall_x(t) \doteq 1$ if and only if $H^{=1} \vdash_I \forall_x t$, which by theorem 1.2 (Pitts' Theorem) holds if and only if $H^{=1} \vdash_I t(X)$, which is equivalent to $\mathfrak{H}(X) \models t(X) \doteq 1$. □

Now we can look at how to determine *within* a given Heyting algebra, whether a given existential formula has a solution in some extension. For this we will first need the following definition:

Definition 4.1. A *primitive existential formula* has the form $\exists x\varphi$, where φ is a quantifier-free conjunction of atomic formulae or their negations. In languages without relation symbols, primitive existential formulae are thus exactly the systems of equations (and inequations) in one variable.

Remark 9. Every equation $t_1 = t_2$ can be expressed in the form $(t_1 \hookrightarrow t_2) \sqcap (t_2 \hookrightarrow t_1) = 1$, so we can w.l.o.g. restrict ourselves to equations of the form $t = 1$.

Theorem 4.3. *Let* $\mathfrak{A} \models T_H$ *and*

$$\exists x \varphi := \exists x (t_1(x) \doteq 1 \wedge ... \wedge t_n(x) \doteq 1 \wedge \neg u_1(x) \doteq 1 \wedge ... \wedge \neg u_m(x) \doteq 1)$$

some primitive existential $L_H(A)$*-sentence. Then* $\exists x \varphi$ *holds in some extension of* $\mathfrak{A}$ *iff the following quantifier-free formulae hold in* $\mathfrak{A}$*:*

$$\exists_x \left(\prod_{i=1}^{n} t_i \right) \doteq 1 \tag{1}$$

$$\neg \forall_x \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \doteq 1 \tag{2}$$

for all $j < m$*.*

Proof. Assume the above formulae hold in $\mathfrak{A}$, then (as (1) holds and by theorem 4.2)

$$\mathfrak{A}(X) / \prod_{i=1}^{n} t_i(X) =: \mathfrak{B}$$

is an extension of $\mathfrak{A}$ in which the formula $t_1(X) \doteq 1 \wedge ... \wedge t_n(X) \doteq 1$ holds[33]. Assume $\varphi(X)$ does not hold in $\mathfrak{B}$, then there is some u_j such that $\mathfrak{B} \models u_j(X) \doteq 1$ and hence

$$\mathfrak{A}(X) \models \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \left[\frac{X}{x} \right] \doteq 1$$

which means

$$\mathfrak{A} \models \forall_x \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \doteq 1,$$

a contradiction to $\mathfrak{A} \models (2)$.

For the converse, let φ be satisfied by some element a in some extension $\mathfrak{B}'$ of $\mathfrak{A}$. Then

$$\mathfrak{B}' \models \prod_{i=1}^{n} t_i \left[\frac{a}{x} \right] \doteq 1$$

and thus

$$\mathfrak{B}' \models \exists_x \left(\prod_{i=1}^{n} t_i \right) \doteq 1.$$

Since $\mathfrak{B}'$ is an extension of $\mathfrak{A}$, we have

$$\mathfrak{A} \models \exists_x \left(\prod_{i=1}^{n} t_i \right) \doteq 1. \qquad (1)$$

[33] Since $t_1 \doteq 1 \wedge ... \wedge t_n \doteq 1$ iff $t_1 \sqcap ... \sqcap t_n \doteq 1$

Assume for some j

$$\mathfrak{A} \models \forall_x \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \doteq 1.$$

Then

$$\mathfrak{B}' \models \forall_x \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \doteq 1$$

and by theorem 1.2 (Pitts' theorem)

$$\mathfrak{B}' \models \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \doteq 1 \left[\frac{a}{x} \right]$$

and therefore $\mathfrak{B}' \models u_j(a)$, a contradiction to $\mathfrak{B}' \models \varphi(a)$. Therefore

$$\mathfrak{A} \models \neg\forall_x \left(\prod_{i=1}^{n} t_i \hookrightarrow u_j \right) \doteq 1 \qquad (2)$$

□

Remark 10. Since in an existentially closed structure any existential formula has a solution iff it has a solution in some extension, the previous theorem yields a method to eliminate the quantifier in primitive existential formulae in e.c. Heyting algebras. As is well known, it follows that e.c. Heyting algebras have quantifier elimination:

Since

1. any formula is equivalent to a formula in prenex normal form (i.e. with all quantifiers at the beginning) with quantifier-free part in disjunctive normal form and

2. existential quantifiers are distributive over disjunctions,

we can inductively eliminate one existential quantifier (and hence universal quantifiers as well) after another in any formula with multiple quantifiers.

It follows that – if we want to axiomatize the class of e.c. Heyting algebras – we can restrict ourselves to the primitive existential formulae:

Corollary 4.1. *The class of existentially closed Heyting algebras is axiomatizable in the language $L_H \cup \{\exists_x, \forall_x\}$ with T_H extended by the (universal closure of the) following formulae:*

$$\exists x \left(\prod_{i=1}^{n} t_i \doteq 1 \wedge \bigwedge_{i=1}^{m} \neg u_i \doteq 1 \right) \leftrightarrow \left(\exists_x \left(\prod_{i=1}^{n} t_i \right) \doteq 1 \wedge \bigwedge_{i=1}^{m} \neg\forall_x \left(\prod_{i=1}^{n} t_i \hookrightarrow u_i \right) \doteq 1 \right)$$

for every finite set of terms $\{t_1(x,\overline{y}), ..., t_n(x,\overline{y}), u_1(x,\overline{y}), ..., u_m(x,\overline{y})\}$*;*

which means the class of T_H-e.c. structures is an elementary class, and since we have already shown that T_H has the amalgamation property, we finally get our intended result:

Corollary 4.2. *The theory of Heyting algebras has a model completion.*

5 Appendix

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Order of connective strength of logical symbols	$\neg, \forall, \exists, \wedge, \vee, \rightarrow, \leftrightarrow, \top, \bot$
Order of connective strength in Heyting algebras	$\sqcap, \sqcup, \hookrightarrow, 1, 0$
Structures	$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, ...$
Their underlying universes	$A, B, C, ...$
Propositional theories	Γ, Δ
Formulae	$\varphi, \psi, \chi, \phi$
φ holds in a structure (or theory) $\mathfrak{A}$	$\mathfrak{A} \models \varphi$
φ is provable from Γ in IpC	$\Gamma \vdash_I \varphi$
φ is an intuitionistic tautology	$\vdash_I \varphi$
$\{\psi\} \vdash_I \varphi$	$\psi \vdash_I \varphi$
The set of propositional variables in φ	$\mathsf{Var}(\varphi)$
Sequents in the sequent calculus for IpC	$\Gamma \succ \varphi$
The sequent $\Gamma \succ \varphi$ is provable in IpC	$\vdash \Gamma \succ \varphi$
Tuple of variables	$\overline{x}$
A formula φ has the free variables $x, \overline{y}$	$\varphi(x, \overline{y})$
Substituting x by some term t in $\varphi(x)$	$\varphi\left[\frac{t}{x}\right]$ or $\varphi(t)$
$\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$	$\mathfrak{A} \prec \mathfrak{B}$
$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent	$\mathfrak{A} \equiv \mathfrak{B}$
$\mathfrak{A}$ extended by constant symbols from $B \subseteq A$	$(\mathfrak{A}, B)$
The language of Heyting algebras	$L_H = (1, 0, \sqcap, \sqcup, \hookrightarrow)$
The theory of Heyting algebras	T_H
The class of models of a theory T	$\mathsf{Mod}(T)$
The propositional formula φ interpreted as L_H-term	$[\varphi]$
The freely generated Heyting algebra over A	$\mathcal{H}_A$
The set of L_H-terms t with $\mathfrak{H} \models t \doteq 1$	$H^{=1}$
The theory of $\mathfrak{A}$ (i.e. $\{\varphi \mid \mathfrak{A} \models \varphi\}$)	$\mathsf{Th}(\mathfrak{A})$
Some language L extended by constant symbols from a set A	$L(A)$

Table 3: List of notations used

Zusammenfassung

Das Ziel dieser Bachelorarbeit ist es, die Existenz einer Modellvervollständigung (*model completion*) der Theorie der Heyting-Algebren zu beweisen. Dazu interpretieren wir diese als algebraische Modelle der intuitionistischen Aussagenlogik (IpC - *intuitionistic propositional calculus*). Dies erlaubt uns, mit dem zentralen Satz aus [Pit92] – welcher besagt, dass jede zweitstufige aussagenlogische Formel in IpC äquivalent zu einer erstufigen ist – zu zeigen, dass die Klasse der existentiell abgeschlossenen Heyting-Algebren eine elementare Klasse ist, woraus die Existenz eines Modellbegleiters (*model companion*) folgt. Dass dieser eine Modellvervollständigung darstellt folgt dann aus der Tatsache, dass die Theorie der Heyting-Algebren die Amalgamationseigenschaft (*amalgamation property*) hat.

In Abschnitt Eins wird intuitionistische Aussagenlogik und der Satz von Pitts vorgestellt. Abschnitt Zwei widmet sich Verbänden (*lattices*) und Heyting-Algebren und das Verhältnis dieser zu IpC wird untersucht. Abschnitt Drei stellt die notwendigen modelltheoretischen Definitionen und Resultate vor, bevor in Abschnitt Vier der zentrale Beweis dieser Arbeit gegeben wird.

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich diese Arbeit selbständig verfasst habe, keine anderen als die angegebenen Quellen/Hilfsmittel verwendet habe und alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen wurden, als solche kenntlich gemacht habe. Darüber hinaus erkläre ich, dass diese Abschlussarbeit nicht, auch nicht auszugsweise, bereits für eine andere Prüfung angefertigt wurde.

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