0 Introduction

Consider the structure $\mathbb{R} = (\mathbb{R}, <, +, -, \cdot, 1)$

**Remark 0.1.** the relation “<” is definable in the structure $(\mathbb{R}, +, 0, -, \cdot, 1)$, because $x < y$ is equivalent to $\exists z (\neg z \not= 0 \land x + z \cdot z \not= y)$

$\text{Th}(\mathbb{R}) = \text{RCF}$ is complete and has quantifier elimination.

$\mathfrak{M} = (M, R_1, R_2, \ldots, \text{Def}(\mathfrak{M})$ is the smallest collection $D$ of subsets of the cartesian product of $M, M^2, \ldots$ s.t. $R_i \in D$ and $D$ is closed under finite unions, finite intersections, taking complements, projections and cartesian products

**Definition 0.1.** An ordered structure $\mathfrak{M} = (M, <, \ldots) \text{ is o-minimal, if every}$

definable subset $X \subseteq M^1 \text{ is a finite union of singletons and open intervals of the form } (a, b) \text{ with } a, b \in M \cup \{-\infty, \infty\}$

Generally we consider only ordered structures where the order is dense and has no endpoints

**Proposition 0.1.** $\mathbb{R}$ is o-minimal

**Proof.** By QE, if $X \subseteq \mathbb{R}^1$ is definable, then $X = \varphi(\mathbb{R})$ for some quantifier-free formula $\varphi(x_0)$

**Example 0.1.**

- $\mathbb{R}_{\text{exp}} = (\mathbb{R}, e^x)$ is o-minimal
- $(\mathbb{R}, <)$ is o-minimal
- $(M, <) \not= \text{DLO}$ is o-minimal
- $(\mathbb{Q}, <, +, -, 0, 1)$ is not o-minimal (Take the set $X = \{x \in \mathbb{Q} | \exists y y^2 = x\}$)

0.1 The Rila-Wilke Theorem

**Definition 0.2.** A point in $\mathbb{R}^n$ all of whose coordinates are rational is called rational point

**Remark 0.2.**

- Algebraic curves:
  - $y = f(x)$ with $f \in \mathbb{Q}(X)$ has many rational points
  - $x^n + y^n = 1$ for $n = 2$ many rational points, for $n > 2$ very few (finite)
• Non-algebraic curves:
  - \( y = e^x \) has only one rational point \((0, 1)\)
  - \( y = 2^x \) has infinitely many rational points: all \((m, 2^n)\) with \(m \in \mathbb{Z}\)

**Definition 0.3** (Height of rational numbers).
For \( x \in \mathbb{Q} \), let \( x = \frac{a}{b} \) with \( a, b \in \mathbb{Z}, (a, b) = 1 \). Define \( h(x) = \max \{|a|, |b|\} \)
For \( x \in \mathbb{Q}^n \), \( x = (x_1, \ldots, x_n) \) let \( h(x) := \max_{i \in \{1, \ldots, n\}} h(x_i) \)

**Remark 0.3.** Let \( r \in \mathbb{Z}_{>0} \), then \( \{x \in \mathbb{Q} \, | \, h(x) \leq r\} \) is finite and has cardinality \( \leq 2r^2 + 1 \)

**Definition 0.4.** Given \( X \subset \mathbb{R}^n \), let \( N(X, r) = |\{x \in X \, | \, x \in \mathbb{Q}^n \text{ and } h(x) \leq r\}| \)

**Example 0.2.**
- If \( X = \mathbb{R} \subset \mathbb{R}^1 \), \( N(X, r) \sim r^2 \). The same holds whenever \( X \) is the graph of a rational function
- If \( X : y = 2^x \), \( N(X, r) \sim \log_2 N(\mathbb{R}, r) \)

**Fact** The probability of two randomly chosen positive integers being relatively prime is \( \frac{6}{\pi^2} \)

\[ \lim_{r \to \infty} \frac{N(\mathbb{R}, r)}{r^2} = \frac{\pi^2}{6} \]

**Exercise 0.1.** \( N(\mathbb{R}, r) = ? \)

**Higher dimensions:** \( X = \{(x, y, z) \in \mathbb{R}^3 \, | \, x^y = z\} \) is non-algebraic, but contains for each \( y \in \mathbb{Q} \) the algebraic set \( \{(x, y, z) \, | \, x^y = z\} = X_y \)

**Definition 0.5.** For \( X \subset \mathbb{R}^n \), let \( X^{alg} \), the algebraic part of \( X \), be the union of all connected, infinite semialgebraic sets contained in \( X \) and let \( X^{tr} \), the transcendental part of \( X \), be \( X \setminus X^{alg} \)

**Theorem 0.1** (Pila-Wilkie). Suppose \( X \subset \mathbb{R}^n \) is definable in o-min structure \((\mathbb{R}, <, \ldots)\), then for every \( \epsilon > 0 \) there is a constant \( c \in \mathbb{R} \) such that \( N(X^{tr}, r) \leq cr^\epsilon \) (subpolynomial)

**Conjecture** (Wilkie). Suppose \( X \subset \mathbb{R}^n \) is definable in \( \mathbb{R}_{\exp} \), then there exist constants \( c_1, c_2 \) s.t. \( N(X^{tr}, r) \leq c_1 (\ln r)^{c_2} \)

**0.2 The (Pila-Zannier proof of the) Manin Mumford Conjecture**

**Theorem 0.2** (Manin-Mumford Conjecture for \((\mathbb{C}^*, \cdot)\)). Suppose \( V \subset (\mathbb{C}^*)^n \) is an algebraic subvariety (For us, an algebraic variety is a subset of some \( \mathbb{C}^n \) defined by a (finite) system of polynomial equations). Then there exist \( b_1, \ldots, b_n \in \mathbb{C} \)
\( \mu^n \) and algebraic subgroups (This means in practice, each \( B_i \) is defined by a system of equations of the form \( x_1^{m_1} \cdot \ldots \cdot x_n^{m_n} = 1, m \in \mathbb{Z} \)) \( B_1, \ldots, B_m \) of \( \mathbb{C}^n \) s.t.

\[
V \cap \mu^n = \bigcup_{i=1}^{m} b_i(B_i \cap \mu^n)
\]

where \( \mu \) denotes the set of roots of unity

\[
\mathbb{R} \rightarrow S^1 \quad t \mapsto e^{\pi it} \\
\mathbb{C} \rightarrow \mathbb{C}^\times \\
z \mapsto e^{iz}
\]

(\( \mathbb{R}, <, \ldots, \exp \)), then \( \{ z \mid \exp(z) = 1 \} = \{ (0, 2\pi k) \mid k \in \mathbb{Z} \} \subset \mathbb{R}^2 \) (not o-minimal)

## 1 O-minimal structures

(following: Speissegger - “O-minimal structures”, Peterzil - “A selfguide to o-minimality”,
van den Dries - “Tame topology and o-minimal structures”)

**Definition 1.1.** \( \mathfrak{M} = (M, <) \) is an ordered structure if \( < \) is a dense linear order without end points on \( M \)

From now on, \( \mathfrak{M} \) is always an ordered structure. This yields the order topology on \( \mathfrak{M} \). The topology with basic open sets \( (a, b) \) with \( a, b \in M \cup \{-\infty, \infty\} \) \( M^n \): the topology with basic open sets \( I = I_1 \times \ldots \times I_n \) where each \( I_i \) is an open interval (\( I \) is an open box).

**Remark 1.1.** If \( M = \mathbb{R} \), then these are the usual topologies on \( \mathbb{R}, \mathbb{R}^2 \ldots \)

**Definition 1.2.** A subset \( S \subset M^n \) is definably connected if there are no definable open sets \( U, V \subset M^n \) such that

- \( S = (S \cap U) \cup (S \cap V) \)
- \( (S \cap U) \cap (S \cap V) = \emptyset \)
- \( S \cap U \) and \( S \cap V \) are non-empty

**Remark 1.2.** If \( S \) is connected, then it is definably connected

**Exercise 1.1.** (1) The image of a definably connected definable set under a definable continuous map is definably connected

(2) Let \( S, T \subset M^n \) be definably connected definable sets with \( \text{cl} S \cap T \neq \emptyset \), then \( S \cup T \) is definably connected

**Definition 1.3.** \( \mathfrak{M} \) is definably complete if every definable subset of \( M \) has an infimum and a supremum in \( M \cup \{-\infty, \infty\} \)

**Exercise 1.2.** Assume \( \mathfrak{M} \) is definably complete

(1) Every interval is definably connected
(2) (Intermediate Value Theorem) Let $f, g : I \to M$ be definable and continuous, with $I \subset M$ an interval. Assume $f(x) \neq g(x), \forall x \in I$

Then: Either $f(x) > g(x), \forall x \in I$ or $f(x) < g(x), \forall x \in I$

**Definition 1.4.** $\mathcal{M}$ is *o-minimal* if every definable subset of $M$ is a finite union of points and intervals

**Remark 1.3.** If $\mathcal{M}$ is o-minimal, then it is definably complete

Assume $\mathcal{M}$ is o-minimal for the rest of the section

**Exercise 1.3.** (1) Every infinite definable subset of $M$ contains an interval

(2) If $A \subset M^{n+1}$ is definable, then \{ $x \in M^n | A_x$ is finite \} is definable ($A_x := \{ a \in M | (a, x) \in A \}$

**Lemma 1.1.** Let $S \subset M$ be definable and $a \in M$, then there exists $\varepsilon > a$ in $M$ such that $(a, \varepsilon) \subset S$ or $(a, \varepsilon) \subset M \setminus S$.

$\mathcal{M} \equiv \mathcal{M}$, $\mathcal{M}$ o-minimal $\Rightarrow \mathcal{M}$ o-minimal

**Definition 1.5.** Let $\mathcal{M}$ be a structure. $\mathcal{M}$ is minimal if every definable subset of $M$ is finite or has finite complement.

**Exercise 1.4.** $\mathcal{M}$ ordered structure, o-minimal. Then the following are equivalent:

(1) Every $\mathcal{M} \equiv \mathcal{M}$ is o-minimal.

(2) For every definable family \{ $X_a | a \in M^k$ \} of finite subsets of $M$, there is $k \in \mathbb{N}$ such that $X_a$ is the union of $\leq k$ points

(3) For every definable family \{ $X_a | a \in M^k$ \} of subsets of $M$ there is $k \in \mathbb{N}$ such that $X_a$ is the union of $\leq k$ points and intervals

1.1 **Monotonicity**

$\mathcal{M}$ o-minimal, $f : I \to M$ a definable function with $I = (a, b)$

**Definition 1.6.** $f$ is *strictly monotone* if $f$ is constant, strictly increasing or strictly decreasing.

For $c \in I$, $f$ is constant/strictly increasing/strictly decreasing/strictly monotone at $c$ if there exist $c_1 < c < c_2$ such that $f |_{(c_1, c_2)}$ is constant/strictly increasing/strictly decreasing/strictly monotone.

**Exercise 1.5.** 1. If $f$ is strictly monotone at every $c \in I$, then $f$ is strictly monotone

2. Assume $f$ is strictly monotone, then there is an open interval $J \supset I$ such that $f |_J$ is continuous.

**Lemma 1.2.** Assume $f(x) > x$ for all $x \in I$, then there exists an open interval $J \supset I$ and $c > J$ such that $f(x) > c$ for all $x \in J$
Proposition 1.1. Let \( S \subset I^2 \) be definable. There exists an open interval \( J \subset I \) such that
\[
\Delta^*(J) := \{(x, y) \in J^2 \mid y > x \}
\]
is either a subset of \( S \) or of \( I^2 \setminus S \).

Remark 1.4. Finite Ramsey: For every \( n \), there is \( N \) such that every 2-coloring of \( [N]^2 \) has a monochromatic set of size \( n \).

Corollary 1.1. Let \( S_1, \ldots, S_k \subset M^2 \) be definable. Assume \( I^2 \subset \bigcup_{i=1}^k S_i \); then there exists \( i \in \{1, \ldots, k\} \) and an open interval \( J \subset I \) such that \( \Delta^*(J) \subset S_i \).

Corollary 1.2. \( f : I \to M \) definable, then there exists \( J \subset I \) such that \( f \mid J \) is strictly monotone.

Theorem 1.1. Monotonicity theorem (\( \mathfrak{M} \)-minimal, \( I = (a,b) \) interval, \( f : I \to M \) definable)
There exist \( k \in \mathbb{N} \) and \( a_1, \ldots, a_k \in I \) such that \( a_0 := a < a_1 < \ldots < a_k < a_{k+1} := b \) and for every \( i \in \{0, \ldots, k\} \), \( f \mid (a_i, a_{i+1}) \) is strictly monotone and continuous.

Proof. Let \( B = \{x \in I \mid f \text{ is strictly monotone and continuous at } x\} \subset I \) (definable).
Claim: \( I \setminus B \) is finite. Thus let \( a_1, \ldots, a_k \) be an enumeration of \( I \setminus B \) and let \( a_0 := a, a_{k+1} := b \). For each \( i \in \{0, \ldots, k\} \), \( f \) is strictly monotone and continuous at every \( x \in (a_i, a_{i+1}) \).
\[ \Rightarrow f \mid (a_i, a_{i+1}) \text{ is strictly monotone and continuous.} \]
Proof of claim: Suppose not, then by \( \mathfrak{M} \)-minimality, \( I \setminus B \) contains an open interval \( J \). By the corollary there is \( J' \subset J \) such that \( f \) is strictly monotone on \( J' \). By Exercise, there is an interval \( J'' \subset J' \) such that \( f \) is continuous in \( J'' \), but then \( J' \subset I \setminus B \subset B' \).

Corollary 1.3. \( f : I = (a,b) \to M \) definable

(i) The limits \( \lim_{x \to a^+} f(x) \), \( \lim_{x \to b^-} f(x) \) and, for every \( c \in (a,b) \), the limits \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) exist in \( M \cup \{-\infty, \infty\} \).

(ii) If \( a, b \in M, g : [a,b] \to M \) definable and continuous, then \( g \) has maximum and minimum in \( [a,b] \).

1.2 Definable Compactness

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Definition 1.7. A definable set \( S \subset M^n \) is definably compact if for every interval \( (a,b) \) in \( M \) and every continuous definable function \( \gamma : (a,b) \to S \subset M^n \), the limits \( \lim_{x \to a^+} \gamma(x) \) and \( \lim_{x \to b^-} \gamma(x) \) belong to \( S \).

Remark 1.5. If \( S \) is compact, then \( S \) is definably compact (?).

Lemma 1.3. If \( S \subset M^n \) is definably compact, then \( \pi_{n-1}(S) \) (projection onto the last \( n-1 \) coordinates) is definably compact.
Proof. Let \((a, b)\) be an interval in \(M\) and \(\gamma : (a, b) \to \pi_{n-1}(S) \subseteq M^{n-1}\) definable. Note: for all \(x \in \pi_{n-1}(S)\), \(S_x := \{y \in M : (x, y) \in S\}\) is a definable subset of \(M\).

Note that by the definable compactness of \(S\), if \((c, d) \subseteq S_x\), then \(c, d \in S_x\), this means that \(S_x\) is closed and bounded. Thus, for every \(x \in \pi_{n-1}(S)\), \((x, \inf S_x) \in S\) Define: \(\gamma' : (a, b) \to S, z \mapsto (\gamma(z), \inf S_z)\). By definable compactness \(\lim_{z \to a^-} \gamma'(z)\) and \(\lim_{z \to b} \gamma'(z)\) are in \(S\). Hence \(\lim_{z \to a^-} \gamma(z)\) and \(\lim_{z \to b} \gamma(z)\) are in \(\pi_{n-1}(S)\)

**Theorem 1.2.** Let \(S\) be a definable subset of \(M^n\). \(S\) is definably compact iff \(S\) is closed and bounded.

**Proof.** “\(\Rightarrow\)” by monotonicity theorem

“\(\Leftarrow\)” Assume \(S\) is definably compact.

- **\(S\) is bounded:** \(S\) is bounded iff the image of \(S\) under every projection onto a single coordinate is bounded. Therefore by Lemma 1.3 we can assume \(n = 1\), which is easy.

- **\(S\) is closed:** By induction on \(n\).

  \(n = 1:\) easy.

  \(n \geq 2:\) Suppose that \((x, y) \in \overline{S} \setminus S\), where \(x \in M^{n-1}, y \in M\).

  * \(S_x := \{z \in M : (x, z) \in S\}\) is closed

  * \(S_x = \overline{S_x}\)

  There is a closed interval \(I\) with \(y \in \text{int}I\) and \(I \cap S_x = \emptyset\). Let \(D \subseteq M^{n-1}\) be a closed box with \(x \in \text{int}D\). Let \(S_1 := S \cup (D \times I)\)

  * \((x, y) \in S_1\)

  * \(x \in \pi_{n-1}(S_1) \setminus \pi_{n-1}(S_1)\), since:

    \[
    (\{x\} \times I) \cap S = \{x\} \times (I \cap S_x) = \emptyset
    \]

  * (IH) \(\pi_{n-1}(S_1)\) is not definably compact

  * \(S_1\) is not definably compact \(\Rightarrow\) \(S\) is not definably compact

\(\square\)

### 1.3 Cells and cell decomposition

(B.A. Seite 21)

**Definition 1.8** (Cells). Let \(\sigma \in \{0, 1\}^n\), let \(\sigma' = \sigma |_{n-1}\). A definable subset \(C \subseteq M^n\) is a \(\sigma\)-cell, if one of the following holds:

(i) \(n = 1, \sigma(0) = 0, C = \{a\}\) for some \(a \in M\)

(ii) \(n = 1, \sigma(0) = 1, C\) is a (non-empty) open interval

(iii) \(n > 1, \sigma(n-1) = 0, C' = \pi_{n-1}\) is a \(\sigma'\)-cell and \(C\) is the graph of a definable function from \(C'\) to \(M\)

(iv) \(n > 1, \sigma(n-1) = 1, C'\) is a \(\sigma'\)-cell and \(C = (f, g)_{C'} := \{(x, y) : x \in C', f(x) < y < g(x)\}\) for some definable \(f, g : C' \to M\)
Lemma 1.4. If $C \subset M^n$ is a cell and $m \leq n$, then $\pi_m(C)$ and for any $a \in \pi_m(C)$ the $C_a = \{y \in M^{n-m} \mid (a, y) \in C\}$ are cells.

Proof. That $\pi_m(C)$ is a cell holds by definition. $\pi^n_m = \pi^{n+1}_m \circ \ldots \circ \pi^n_{m-1} \circ \pi^n_{m-1}, C = (f_{n-1}, g_{n-1}) \pi^n_{m-1}(C), \pi^n_{m+1}(C) = (f_{m+1}, g_{m+1}) \pi^n_{m-1}(C)$

$C_a$ is the cell obtained as follows:

- $C_0 = (f_m(a), g_m(a))$
- $C_1 = (f_{m+1}(a, x), g_{m+1}(a, x))C_0 \subset M^2$
- $\vdots$
- $C_{n-m-1} = (f_{n-1}(a, x), g_{n-1}(a, x))C_{n-m-2} \subset M^{n-m}$

$1 \leq m \leq n$ let $\pi^n_m$ be the projection onto the first $m$ coordinates. If $\varepsilon : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ is strictly increasing $\pi_i : M^n \rightarrow M^n, (x_1, \ldots, x_n) \rightarrow (x_{\varepsilon(1)}, \ldots, x_{\varepsilon(n)})$

Definition 1.9. Let $C \subset M^n$ be a $\sigma$-cell

1. $C$ is open if $\sigma(i) = 1, \forall i \in n$
2. $\sum \sigma = \sum_{i=0}^{n-1} \sigma(i)$
3. Let $\tau : \{1, \ldots, \sum \sigma\} \rightarrow \{1, \ldots, n\}$ be strictly increasing enumeration of the elements $i \in \{1, \ldots, n\}$ such that $\sigma(i-1) = 1$

Lemma 1.5. A cell $C$ is an open cell iff it is an open set

Lemma 1.6. Let $C \subset M^n$ be a cell. Then $C_\sigma = \Pi_{i\in\sigma}(C) \subset M^{\sum \sigma}$ is an open cell and $\pi_{i\sigma}|C : C \rightarrow C^n$ is a definable homeomorphism

Proposition 1.2. Every cell in $M^n$ is definably connected

Definition 1.10. (1) Let $C$ be a finite collection of cells in $M^n$ and $U \subset M^n$. $C$ is a cell decomposition of $U$ if $C$ is a partition of $U$ and, if $n \geq 2$, the set $\Pi_{n-1}(C) := \{\Pi_{n-1}(C) \mid C \in C\}$ is a cell decomposition of $\Pi_{n-1}(U)$

(2) If $Z \subset U$, then $U$ is compatible with $Z$ if for every cell $C \in C$ either $C \subset Z$ or $C \cap Z = \emptyset$

(3) If $C$ and $D$ are cell decompositions of $U$, we say that $D$ is a refinement of $C$, if $D$ is compatible with every $C \in C$

Example 1.1. Consider the set $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ (13 cells for all of $\mathbb{R}^2$)

Remark 1.6. Let $C$ be a cell decomposition of $U \subset M^{n+m}$ and let $x \in M^n$. Then $C_x = \{C_x \mid C \in C\}$ is a cell decomposition of $U_x = \{y \in M^m \mid (x, y) \in U\}$

Proof. by induction on $m$
Let $Z_1, \ldots, Z_k \subset M^n$ and let $\mathcal{B}$ be the boolean algebra generated by them. Then: a cell decomposition of $M^n$ is compatible with the $Z_i$ iff it is compatible with all atoms of $\mathcal{B}$

Proof. Every atom $B$ of $\mathcal{B}$ has the form $B = B_1 \cap \ldots \cap B_k$, where each $B_i$ is either $Z_i$ or $M^n \setminus Z_i$.

$\Leftarrow$ If $C$ is compatible with the $Z_i$, for each atom $B$ of $\mathcal{B}$

- if $C$ contains every $B_i$, then $C$ is contained in $B$
- if $C$ is disjoint for some $B_i$, then $C$ is disjoint from $B$

$\Rightarrow$ If $C$ is compatible with the $Z_i$, for each atom $B$ of $\mathcal{B}$

- if $C \cap B_i$ is open, then $C \cap B_i$ is open
- if $C \cap B_i$ is closed, then $C \cap B_i$ is closed

Theorem 1.3 (Cell Decomposition Theorem). (Siehe "Zellzerlegung O-minimalen Strukturen")

(I)_n Let $S_1, \ldots, S_k \subset M^n$ be definable. Then there is a cell decomposition of $M^n$ that is compatible with every $S_i$

(II)_n Let $f : S \to M$ be definable with $S \subset M^n$ definable. Then there is a cell decomposition $\mathcal{C}$ of $M^n$ compatible with $S$ such that for every $C \in \mathcal{C}$, $f \restr C$ is continuous.

Proof. By induction

$n = 1$ $\mathcal{I}_n$ follows easily from the definition of o-minimality, $\mathcal{II}_n$ is the monotonicity theorem.

$n > 1$

Lemma 1.7. Let $S \subset M^n$ be definable. The following are equivalent

1. $S$ is sparse if (Definition) $\text{int}(S) = \emptyset$ (Remark: A cell is sparse iff it is not open)
2. The set $S' := \{ x \in M^{n-1} \mid S_x \text{ is infinite} \}$ is sparse
3. $S$ is nowhere dense, if (Definition) $\text{int}(S) = \emptyset$

In particular, a finite union of sparse subsets of $M^n$ is sparse.

Proof $\Rightarrow 2$ Assume $S'$ is not sparse. Then we can find an open box $U \subset S'$. For any $x \in U$, since $S_x \subset M$ is infinite, $S_x$ contains an interval.

Fix a decomposition of $S_x$ as a union of finitely many open intervals and points. Let $I_x := \text{the first open interval in the decomposition}$.

\[ i_x := \begin{cases} \inf I_x & \text{if } \inf I_x \in M \\ \text{a point in } I_x & \text{otherwise} \end{cases} \]

\[ S_x := \begin{cases} \sup I_x & \text{if } \sup I_x \in M \\ \text{a point in } I_x \text{ greater than } i_x & \text{otherwise} \end{cases} \]

$i_x$ and $S_x$ are definable functions on $U$. By $\mathcal{II}_{n-1}$, there is a cell decomposition $\mathcal{C}$ of $M^{n-1}$ with $U$ such that $i_x \restr C$ and $S_x \restr C$ are continuous for any $C \in \mathcal{C}$. Then there is an open cell $C' \in \mathcal{C}$ contained in $U$ (whence $(i_x \restr C', S_x \restr C') \restr C'$ is an open cell and is contained in $S$)
1.4 The Pila-Wilkie Theorem

**Theorem 1.4.** Let $\mathcal{R} = (\mathbb{R}, <, +, -, 0, 1, ...)$ be an o-minimal expansion of the real field. Let $X \subset \mathbb{R}^n$ be definable in $\mathcal{R}$. Then for every $\varepsilon > 0$, there is $c = c(X, \varepsilon) > 0$ such that $\forall T \geq 1 \ N(X^{tr}, T) \leq cT^\varepsilon$, where

- $X^{tr} = X \setminus X^{alg}$, $X^{alg} = \bigcup \{Y \mid Y \subset X \text{ is infinite, connected, semialgebraic}\}$
- For $X \subset \mathbb{R}^n$, $N(X, T) = |X(\mathbb{Q}, T)| := \{|x \in X \mid x \in \mathbb{Q}^n \wedge H(x) \leq T\}$

For $q \in \mathbb{Q}^n$, $H(q) = \max(q_j)$, for $q \in \mathbb{Q}^n$, $H(q) := \max(|a|, |b|)$, if $q = \frac{a}{b}$, $\gcd(a, b) = 1$, $H(0) = 0$.

**Theorem 1.5** (Uniform Pila-Wilkie). Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +, -, 0, 1)$. Let $(X_b)_b$ be a definable family of subsets of $\mathbb{R}^n$. For every $\varepsilon > 0$, there exists a family $(Y_b)_b$ and $c = c((X_b)_b, \varepsilon)$ such that for every $b \in B$

- $Y_b \subset X^{alg}_b$
- $\forall T \geq 1, N(X_b \setminus Y_b, T) \leq cT^\varepsilon$

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**Definition 1.11.** A hypersurface of degree $d$ in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n \mid f(x) = 0\}$ where $f \in \mathbb{R}[X]$ has degree $d$.

**Definition 1.12.** Let $X \subset \mathbb{R}^n, k \in \mathbb{Z}_+$. A partial $k$-parametrisation of $X$ is a function $f : (0, 1)^{\dim X} \to X$ such that $\forall \alpha \in \mathbb{N}^{\dim X}$ with $|\alpha| \leq k$ ($|\alpha| = \sum \alpha_i$):

- $f^{(\alpha)}$ is continuous
- $|f^{(\alpha)}|_{\max|y|} \leq 1, \forall x \in (0, 1)$

**Definition 1.13.** A $k$-parametrisation of $X$ is a finite set $S$ of partial parametrisations of $X$ such that $\bigcup_{f \in S} \operatorname{im}(f) = X$
Theorem 1.6 (Bombieri-Pila). Given $0 < m < n, d > 0$, there exist $k = k(m, n, d) \in \mathbb{Z}_+$, $c = c(m, n, d) > 0$ such that if $f : (0, 1)^m \to X$ is a $k$-parametrisation of its image, then $X(\mathbb{Q}, T) \cup$ union of $\leq cT^\varepsilon$ hypersurfaces of degree $d$. Moreover, $\varepsilon \to 0$ as $d \to \infty$.

Let $\mathcal{M}$ be an o-minimal expansion of a real closed field $M$.

**Definition 1.14.**
- An element $a \in M$ is strongly bounded if there is $N \in \mathbb{N}$ such that $|a| \leq N$.
- $a \in M^n$ is strongly bounded if there is $N \in \mathbb{N}$ such that $|a| := \max |a_i| \leq N$.
- A subset $A$ of $M^n$ is strongly bounded if there is $N \in \mathbb{N}$ such that $|a| \leq N \forall a \in A$.

**Theorem 1.7** (Parametrisation Theorem). Let $X$ be a strongly bounded definable subset of $M^n$, for every $k \in \mathbb{Z}_+$, there exists a definable $k$-parametrisation of $X$.

**Corollary 1.4.** Let $m, r \geq 1, X \subset (0, 1)^m$ definable. Then there exists a finite set $S$ of functions $(0, 1)^{dimX} \to \mathbb{R}^d$ of class $C^r$ such that $\bigcup_{\phi \in S} \text{im}(\phi) = X$ and $\|\phi_i(x)\| \leq 1$ for all $\phi \in S, \alpha \in \mathbb{N}^{dimX}$ with $|\alpha| \leq r$ and $x \in (0, 1)^{dimX}$.

**Definition 1.15.** A partial $r$-parametrisation of $X$ is a $C^r$-function $f : (0, 1)^{dimX} \to X$ such that $\forall \alpha \in \mathbb{N}^{dimX}$ with $|\alpha| \leq r, \forall x |f^{(\alpha)}(x)| \leq N$ for some $N \in \mathbb{N}$.

**Proof of corollary.** Let $S$ be an $r$-parametrisation of $X$. Cover $(0, 1)^{dimX}$ with $N^{dimX}$ cubes of side $\frac{1}{k}$ and for each cube $K$, let $\lambda_K : (0, 1)^{dimX} \to K$ be the obvious bijection. Let $S := \{\phi \circ \lambda_K | \phi \in S^K, K$ is one of the cubes}, $d := dimX$.

$$\mathbb{R}^d \xrightarrow{\lambda} \mathbb{R}^d \xrightarrow{\phi} \mathbb{R}^n.$$  

$$\frac{\partial(\phi \circ \lambda)}{\partial x_i}(x) = \frac{\partial \phi}{\partial \lambda}(\lambda(x)) \cdot \frac{\partial \lambda}{\partial x_i}(x) = \frac{1}{N} \frac{\partial \phi}{\partial x_i}(\lambda(x))$$

$$\frac{\partial(\phi \circ \lambda)}{\partial^{x_1 \ldots x_d}}(x) = \left( \frac{1}{N} \right)^{|\alpha|} \frac{\partial \phi}{\partial^{x_1 \ldots x_d}} \leq \frac{1}{N} \cdot N.$$

**Corollary 1.5** (Uniform parametrisation Theorem). Let $n, m, r \geq 1, X \subset (0, 1)^{m \times n}$ definable. Then there exist $N \in \mathbb{N}$ and for all $y \in M^n$ a finite set $S_y$ of $N C^r$-functions $(0, 1)^{dimX_y} \to X_y$ such that (1) $\bigcup_{\phi \in S_y} \text{im}(\phi) = X_y$ and (2) $|\phi(x)| \leq 1$ for all $y, \phi \in S_y, \alpha \in \mathbb{N}^{dimX_y}$ with $|\alpha| \leq r$ and $x \in (0, 1)^{dimX_y}$.

**Proof.** Suppose not, i.e. for every $N \in \mathbb{N}$ there is $y_N \in M^n$ such that ... but this implies that there is $y \in M^n$ such that “Parametrisation theorem does not hold for $X_y$.”

For $N \in \mathbb{Z}_+$, let $\Gamma_N(v)$ be the set of formulas expressing “for every set of $N$ functions satisfying (2), the union of their images is not $X_v$.” Let $\Gamma(v) = \bigcup_N \Gamma_N(v)$.

$\Gamma(v)$ is finitely satisfiable. Compactness: there is $N > M$ and $y \in N$ such that $N = \Gamma(y)$.

Main Lemma - siehe Skript.
Theorem 1.8 (Laurent ’80s, conjectured by Lang for curves, “Manin Mumford Conjecture for $\mathbb{C}^m$” (multiplicative group). Suppose $V \subset (\mathbb{C}^*)^d$ is an irreducible subvariety. Then there are finitely many algebraic subgroups $B_1, ..., B_n$ of $(\mathbb{C}^*)^d$ and $b_1, ..., b_n \in (\mathbb{C}^*)^d$ such that

- $b_i B_i \subset V$ for all $i$
- $V \cap \mu^d = \bigcup_i b_i (B_i \cap \mu^d)$ (where $\mu^d = \text{Tor}((\mathbb{C}^*)^d)$ and $(B_i \cap \mu^d) = \text{Tor}(B_i)$ and $\mu := \text{roots of unity} \subset \mathbb{C}^*$)

**Fact:** Every algebraic subgroup of $(\mathbb{C}^*)^d$ is defined by a finite set of equations of the form:

$y_1^{m_{11}} ... y_n^{m_{1n}} = 1, m_i \in \mathbb{Z}$

$y_1^{m_{k1}} ... y_n^{m_{kn}} = 1$

and $\dim B = d - \text{rk}_\mathbb{Q}(m_{ij})$. $LB = \{x \mid M \cdot x = 0\} \subset \mathbb{C}^d$. $\dim LB = \dim B$. In particular,

- If $V$ contains no cosets of infinite algebraic subgroups, then $V \cap \mu^d$ is finite

**Proof.** See “The case of Tori” onwards. $(\mathbb{C}^*)^d = G_m^d$