# Vorlesungsmitschrift Model theory and applications

O-minimality and diophantine geometry

Aktueller Stand: 15. Juli 2013
bei Juan-Diego Caycedo

# 0 Introduction

Consider the structure $\overline{\mathbb{R}} = (\mathbb{R}, <, +, 0, -, \cdot, 1)$

**Remark 0.1.** the relation "$<$" is definable in the structure $(\mathbb{R}, +, 0, -, \cdot, 1)$, because $x < y$ is equivalent to $\exists z(\neg z \doteq 0 \wedge x + z \cdot z \doteq y)$

$\mathsf{Th}(\overline{\mathbb{R}}) = \mathsf{RCF}$ is complete and has quantifier elimination.

$\mathfrak{M} = (M, R_1, R_2, ...,,$ $Def(\mathfrak{M})$ is the smallest collection $D$ of subsets of the cartesian product of $M, M^2, ...$ s.t. $R_i \in D$ and $D$ is closed under finite unions, finite intersections, taking complements, projections and cartesian products

**Definition 0.1.** An ordered structure $\mathfrak{M} = (M, <, ...)$ is *o-minimal*, if every definable subset $X \subset M^1$ is a finite union of singletons and open intervals of the form $(a, b)$ with $a, b \in M \cup \{-\infty, \infty\}$
Generally we consider only ordered structures where the order is dense and has no endpoints

**Proposition 0.1.** $\overline{\mathbb{R}}$ *is o-minimal*

*Proof.* By QE, if $X \subset \mathbb{R}^1$ is definable, then $X = \varphi(\overline{\mathbb{R}})$ for some quantifier-free formula $\varphi(x_0)$ □

**Example 0.1.**

- $\mathbb{R}_{exp} = (\overline{\mathbb{R}}, e^x)$ is o-minimal
- $(\mathbb{R}, <)$ is o-minimal
- $(M, <) \models \mathsf{DLO}$ is o-minimal
- $(\mathbb{Q}, <, +, \cdot, -, 0, 1)$ is *not* o-minimal (Take the set $X = \{x \in \mathbb{Q} \mid \exists y\ y^2 = x\}$)

## 0.1 The Rila-Wilke Theorem

**Definition 0.2.** A point in $\mathbb{R}^n$ all of whose coordinates are rational is called *rational point*

**Remark 0.2.**

- Algebraic curves:
  - $y = f(x)$ with $f \in \mathbb{Q}(X)$ has many rational points
  - $x^n + y^n = 1$ for $n = 2$ many rational points, for $n > 2$ very few (finite)

- Non-algebraic curves:
  - $y = e^x$ has only one rational point $(0,1)$
  - $y = 2^x$ has infinitely many rational points: all $(m, 2^m)$ with $m \in \mathbb{Z}$

**Definition 0.3** (Height of rational numbers)**.**
For $x \in \mathbb{Q}, x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $(a,b) = 1$. Define $h(x) = \max\{|a|, |b|\}$
For $x \in \mathbb{Q}^n$, $x = (x_1, ..., x_n)$ let $h(x) := \max\limits_{i \in \{1,...,n\}} h(x_i)$

**Remark 0.3.** Let $r \in \mathbb{Z}_{>0}$, then $\{x \in \mathbb{Q} \mid h(x) \leq r\}$ is finite and has cardinality $\leq 2r^2 + 1$

**Definition 0.4.** Given $X \subset \mathbb{R}^n$, let $N(X, r) = |\{x \in X \mid x \in \mathbb{Q}^n \text{ and } h(x) \leq r\}|$

**Example 0.2.**

- If $X = \mathbb{R} \subset \mathbb{R}^1$. $N(X, r) \sim r^2$. The same holds whenever $X$ is the graph of a rational function
- If $X : y = 2^x$, $N(X, r) \sim \log_2 N(\mathbb{R}, r)$

**Fact** The probability of two randomly chosen positive integers being relatively prime is $\frac{6}{\pi^2}$
$\Rightarrow \lim\limits_{r \to \infty} \dfrac{N(\mathbb{R}, r)}{r^2} = \dfrac{?}{r^2}$

**Exercise 0.1.** $N(\mathbb{R}, r) =?$

**Higher dimensions:** $X = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^y = z\right\}$ is non-algebraic, but contains for each $y \in \mathbb{Q}$ the algebraic set $\left\{(x, \underbrace{y}_{\in \mathbb{Q}}, z) \mid x^y = z\right\} = X_y$

**Definition 0.5.** For $X \subset \mathbb{R}^n$, let $X^{alg}$, the *algebraic part of* $X$, be the union of all connected, infinite semialgebraic sets contained in $X$ and let $X^{tr}$, the *transcendental part of* $X$, be $X \setminus X^{alg}$

**Theorem 0.1** (Pila-Wilkie)**.** *Suppose $X \subset \mathbb{R}^n$ is definable in o-min structure $(\mathbb{R}, <, ...)$, then for every $\varepsilon > 0$ there is a constant $c \in \mathbb{R}$ such that $N(X^{tr}, r) \leq cr^\varepsilon$ (subpolynomial)*

**Conjecture** (Wilkie)**.** *Suppose $X \subset \mathbb{R}^n$ is definable in $\mathbb{R}_{exp}$, then there exist constants $c_1, c_2$ s.t. $N(X^{tr}, r) \leq c_1(\ln r)^{c_2}$*

## 0.2 The (Pila-Zannier proof of the) Manin Mumford Conjecture

**Theorem 0.2** (Manin-Mumford Conjecture for $(\mathbb{C}^\times, \cdot)$)**.** *Suppose $V \subset (\mathbb{C}^\times)^n$ is an algebraic subvariety (For us, an algebraic variety is a subset of some $\mathbb{C}^n$ defined by a (finite) system of polynomial equations). Then there exist $b_1, ..., b_n \in$*

*$\mu^n$ and algebraic subgroups (This means in practice, each $B_i$ is defined by a system of equations of the form $x_1^{m_1} \cdot ... \cdot x_n^{m_n} = 1$, $m \in \mathbb{Z}$) $B_1, ..., B_m$ of $\mathbb{C}^n$ s.t.*

$$V \cap \mu^n = \bigcup_{i=1}^{m} b_i(B_i \cap \mu^n)$$

*where $\mu$ denotes the set of roots of unity*

$\mathbb{R} \to S^1$
$t \mapsto e^{\tau it}$
$\mathbb{C} \to \mathbb{C}^\times$
$z \mapsto e^{iz}$
$(\mathbb{R}, <, ..., \exp)$, then $\{z \mid \exp(z) = 1\} = \{(0, 2\pi k) \mid k \in \mathbb{Z}\} \subset \mathbb{R}^2$ (not o-minimal)

# 1 O-minimal structures

(following: Speissegger - "O-minmal structures", Peterzil - "A selfguide to o-minimality",
van den Dries - "Tame topology and o-minimal structures")

**Definition 1.1.** $\mathfrak{M} = (M, <)$ is an *ordered structure* if $<$ is a dense linear order without end points on $M$

From now on, $\mathfrak{M}$ is always an ordered structure. This yields the order topology on $\mathfrak{M}$: The topology with basic open sets $(a, b)$ with $a, b \in M \cup \{-\infty, \infty\}$
$M^n$: the topology with basic open sets $I = I_1 \times ... \times I_n$ where each $I_i$ is an open interval ($I$ is an *open box*).

**Remark 1.1.** If $M = \mathbb{R}$, then these are the usual topologies on $\mathbb{R}, \mathbb{R}^2$...

**Definition 1.2.** A subset $S \subset M^n$ is *definably connected* if there are no definable open sets $U, V \subset M^n$ such that

- $S = (S \cap U) \cup (S \cap V)$
- $(S \cap U) \cap (S \cap V) = \emptyset$
- $S \cap U$ and $S \cap V$ are non-empty

**Remark 1.2.** If $S$ is connected, then it is definably connected

**Exercise 1.1.** (1) The image of a definably connected definable set under a definable continuous map is definably connected

(2) Let $S, T \subset M^n$ be definably connected definable sets with $clS \cap T \neq \emptyset$, then $S \cup T$ is definably connected

**Definition 1.3.** $\mathfrak{M}$ is *definably complete* if every definable subset of $M$ has an infimum and a supremum in $M \cup \{-\infty, \infty\}$

**Exercise 1.2.** Assume $\mathfrak{M}$ is definably complete

(1) Every interval is definably connected

(2) (Intermediate Value Theorem) Let $f, g : I \to M$ be definable and continuous, with $I \subset M$ an interval. Assume $f(x) \neq g(x),\ \forall x \in I$
Then: Either $f(x) > g(x), \forall x \in I$ or $f(x) < g(x), \forall x \in I$

**Definition 1.4.** $\mathfrak{M}$ is *o-minimal* if every definable subset of $M$ is a finite union of points and intervals

**Remark 1.3.** If $\mathfrak{M}$ is o-minimal, then it is definably complete

Assume $\mathfrak{M}$ is o-minimal for the rest of the section

**Exercise 1.3.** (1) Every infinite definable subset of $M$ contains an interval

(2) If $A \subset M^{n+1}$ is definable, then $\{x \in M^n \mid A_x \text{ is finite}\}$ is definable ($A_x := \{a \in M \mid (a,x) \in A\}$

**Lemma 1.1.** *Let $S \subset M$ be definable and $a \in M$, then there exists $\varepsilon > a$ in $M$ such that $(a, \varepsilon) \subset S$ or $(a, \varepsilon) \subset M \setminus S$.*

$\mathfrak{N} \equiv \mathfrak{M}$, $\mathfrak{M}$ o-minimal $\Rightarrow \mathfrak{N}$ o-minimal

**Definition 1.5.** Let $\mathfrak{M}$ be a structure. $\mathfrak{M}$ is minimal if every definable subset of $M$ is finite or has finite complement.

**Exercise 1.4.** $\mathfrak{M}$ ordered structure, o-minimal. Then the following are equivalent:

(1) Every $\mathfrak{N} \equiv \mathfrak{M}$ is o-minimal.

(2) For every definable family $\left\{X_a \mid a \in M^k\right\}$ of finite subsets of $M$, there is $k \in \mathbb{N}$ such that $X_a$ is the union of $\leq k$ points

(3) For every definable family $\left\{X_a \mid a \in M^k\right\}$ of subsets of $M$ there is $k \in \mathbb{N}$ such that $X_a$ is the union of $\leq k$ points and intervals

## 1.1 Monotonicity

$\mathfrak{M}$ o-minimal, $f : I \to M$ a definable function with $I = (a, b)$

**Definition 1.6.** $f$ is *strictly monotone* if $f$ is constant, strictly increasing or strictly decreasing.
For $c \in I$, $f$ is *constant/strictly increasing/strictly decreasing/strictly monotone at $c$* if there exist $c_1 < c < c_2$ such that $f\mid_{(c_1,c_2)}$ is constant/strictly increasing/strictly decreasing/strictly monotone.

**Exercise 1.5.**
1. If $f$ s strictly monotone at every $c \in I$, then $f$ is strictly monotone
2. Assume $f$ is strictly monotone, then there is an open interval $J \supset I$ such that $f\mid_J$ is continuous.

**Lemma 1.2.** *Assume $f(x) > x$ for all $x \in I$, then there exists an open interval $J \supset I$ and $c > J$ such that $f(x) > c$ for all $x \in J$*

**Proposition 1.1.** *Let $S \subset I^2$ be definable. There exists an open interval $J \subset I$ such that*

$$\Delta^>(J) := \left\{(x,y) \in J^2 \mid y > x\right\}$$

*is either a subset of $S$ or of $I^2 \setminus S$*

**Remark 1.4.** *Finite Ramsey:* For every $n$, there is $N$ such that every 2-coloring of $[N]^2$ has a monochromatic set of size $n$

**Corollary 1.1.** *Let $S_1, ..., S_k \subset M^2$ be definable. Assume $I^2 \subset \bigcup_{i=1}^{k} S_i$, then there exists $i \in \{1, ..., k\}$ and an open interval $J \subset I$ such that $\Delta^>(J) \subset S_i$*

**Corollary 1.2.** *$f : I \to M$ definable, then there exists $J \subset I$ such that $f \mid_J$ is strictly monotone*

**Theorem 1.1.** ***Monotonicity theorem*** *($\mathfrak{M}$ o-minimal, $I = (a,b)$ interval, $f : I \to M$ definable)*
*There exist $k \in \mathbb{N}$ and $a_1, ..., a_k \in I$ such that $a_0 := a < a_1 < ... < a_k < a_{k+1} := b$ and for every $i \in \{0, ..., k\}$, $f \mid_{(a_i, a_{i+1})}$ is strictly monotone and continuous.*

*Proof.* Let $B = \{x \in I \mid f$ is strictly monotone and continuous at $x\} \subset I$ (definable).
Claim: $I \setminus B$ is finite. Thus let $a_1, ..., a_k$ be an enumeration of $I \setminus B$ and let $a_0 := a, a_{k+1} := b$. For each $i \in \{0, ..., k\}$, $f$ is strictly monotone and continuous at every $x \in (a_i, a_{i+1})$
$\Rightarrow f \mid_{(a_i, a_{i+1})}$ is strictly monotone and continuous.
*Proof of claim:* Suppose not, then by o-minimality, $I \setminus B$ contains an open interval $J$. By the corollary there is $J' \subset J$ such that $f$ is strictly monotone on $J'$. By Exercise, there is an Intervall $J'' \subset J'$ such that $f$ is continuous in $J''$, but then $J' \subset I \setminus B \subset B$↯ □

**Corollary 1.3.** *$f : I = (a,b) \to M$ definable*

*(i) The limits $\lim_{x \to a^+} f(x)$, $\lim_{x \to b^-} f(x)$ and, for every $c \in (a,b)$, the limits $\lim_{x \to c^-} f(x)$, $\lim_{x \to c^+} f(x)$ exist in $M \cup \{-\infty, \infty\}$*

*(ii) If $a, b \in M, g : [a,b] \to M$ definable and continuous, then $g$ has maximum and minimum in $[a,b]$*

## 1.2 Definable Compactness

Siehe Bachelorarbeit S.18

**Definition 1.7.** A definable set $S \subset M^n$ is *definably compact* if for every interval $(a,b)$ ind $M$ and every continuous definable function $\gamma : (a,b) \to S \subset M^n$, the limits $\lim_{x \to a^+} \gamma(x)$ and $\lim_{x \to b^-} \gamma(x)$ belong to $S$

**Remark 1.5.** If $S$ is compact, then $S$ is definably compact (?)

**Lemma 1.3.** *If $S \subset M^n$ is definably compact, then $\pi_{n-1}(S)$ (projection onto the last $n-1$ coordinates) is definably compact*

*Proof.* Let $(a,b)$ be an interval in $M$ and $\gamma : (a,b) \to \pi_{n-1}(S) \subset M^{n-1}$ definable. Note: for all $x \in \pi_{n-1}(S)$, $S_x := \{y \in M : (x,y) \in S\}$ is a definable subset of $M$. Note that by the definable compactness of $S$, if $(c,d) \subset S_x$, then $c,d \in S_x$, this means that $S_x$ is closed and bounded. Thus, for every $x \in \pi_{n-1}(S)$, $(x, \inf S_x) \in S$ Define: $\gamma' : (a,b) \to S,\ z \mapsto (\gamma(z), \inf S_{\gamma(z)})$. By definable compactness $\lim_{z\to a^-} \gamma'(z)$ and $\lim_{z\to b^+} \gamma'(z)$ are in $S$. Hence $\lim_{z\to a^-} \gamma(z)$ and $\lim_{z\to b^+} \gamma(z)$ are in $\pi_{n-1}(S)$

□

**Theorem 1.2.** *Let $S$ be a definable subset of $M^n$. $S$ is definably compact iff $S$ is closed and bounded.*

*Proof.* "$\Leftarrow$" by monotonicity theorem
"$\Rightarrow$" Assume $S$ is definably compact.

- *$S$ is bounded*: $S$ is bounded iff the image of $S$ under every projection onto a single coordinate is bounded. Therefore by Lemma 1.3 we can assume $n = 1$, which is easy.

- *$S$ is closed*: By induction on $n$.

$n = 1$: easy.

$n \geq 2$: Suppose that $(x,y) \in \overline{S} \setminus S$, where $x \in M^{n-1}, y \in M$.

  * $S_x := \{z \in M \mid (x,z) \in S\}$ is closed
  * $S_x = \overline{S_x}$
    There is a closed interval $I$ with $y \in \mathsf{int} I$ and $I \cap S_x = \emptyset$. Let $D \subset M^{n-1}$ be a closed box with $x \in \mathsf{int} D$. Let $S_1 := S \cup (D \times I)$
  * $(x,y) \in \overline{S_1}$
  * $x \in \overline{\pi_{n-1}(S_1)} \setminus \pi_{n-1}(S_1)$, since:

$$(\{x\} \times I) \cap S = \{x\} \times \underbrace{(I \cap S_x)}_{=\emptyset}$$

  * (IH) $\pi_{n-1}(S_1)$ is not definably compact
  * $S_1$ is not definably compact $\Rightarrow$ $S$ is not definably compact

□

## 1.3 Cells and cell decomposition

(B.A. Seite 21)

**Definition 1.8** (Cells)**.** Let $\sigma \in \{0,1\}^n$, let $\sigma' := \sigma\mid_{n-1}$. A definable subset $C \subset M^n$ is a $\sigma$-cell, if one of the following holds:

(i) $n = 1$, $\sigma(0) = 0$, $C = \{a\}$ for some $a \in M$

(ii) $n = 1$, $\sigma(0) = 1$, $C$ is a (non-empty) open interval

(iii) $n > 1$, $\sigma(n-1) = 0$, $C' = \pi_{n-1}$ is a $\sigma'$-cell and $C$ is the graph of a definable function from $C'$ to $M$

(iv) $n > 1$, $\sigma(n-1) = 1$, $C'$ is a $\sigma'$-cell and $C = (f,g)_{C'} := \{(x,y) \mid x \in C', f(x) < y < g(x)\}$ for some definable $f,g : C' \to M$

**Lemma 1.4.** *If $C \subset M^n$ is a cell and $m \le n$, then $\pi_m(C)$ and for any $a \in \pi_m(C)$ the $C_a = \{y \in M^{n-m} \mid (a,y) \in C\}$ are cells.*

*Proof.* That $\pi_m(C)$ is a cell holds by definition.
$\pi^n_m = \pi^{m+1}_m \circ ... \circ \pi^{n-1}_{n-2} \circ \pi^n_{n-1}$. $C = (f_{n-1}, g_{n-1})_{\pi^n_{n-1}(C)}$, $\pi^n_{m+2}(C) = (f_{m+1}, g_{m+1})_{\pi^n_{m+1}(C)}$
... $\pi^n_{m+1}(C) = (f_m, g_m)_{\pi^n_m(C)}$, $a \in \pi^n_m(C)$
$C_a$ is the cell obtained as follows:

- $C_0 = (f_m(a), g_m(a))$
- $C_1 = (f_{m+1}(a,x), g_{m+1}(a,x))_{C_0} \subset M^2$
  ⋮
- $C_{n-m-1} = (f_{n-1}(a,x), g_{n-1}(a,x))_{C_{n-m-2}} \subset M^{n-m}$

□

$1 \le m \le n$ let $\pi^n_m$ be the projection onto the first $m$ coordinates. If $\iota : \{1,...,m\} \to \{1,...,n\}$ is strictly increasing $\pi_\iota : M^n \to M^n$, $(x_1,...,x_n) \mapsto (x_{\iota_1},...,x_{\iota_m})$

**Definition 1.9.** Let $C \subset M^n$ be a $\sigma$-cell

(1) $C$ is *open* if $\sigma(i) = 1,\ \forall i \in n$

(2) $\sum \sigma := \sum_{i=0}^{n-1} \sigma(i)$

(3) Let $\iota_\sigma : \{1,...,\sum\sigma\} \to \{1,...,n\}$ be strictly increasing enumeration of the elements $i \in \{1,...,n\}$ such that $\sigma(i-1) = 1$

**Lemma 1.5.** *A cell $C$ is an open cell iff it is an open set*

**Lemma 1.6.** *Let $C \subset M^n$ be a cell. Then $C_\sigma := \prod_{\iota_\sigma}(C) \subset M^{\Sigma\sigma}$ is an open cell and $\pi_{\iota_\sigma}|_C : C \to C^\sigma$ is a definable homeomorphism*

**Proposition 1.2.** *Every cell in $M^n$ is definably connected*

**Definition 1.10.** (1) Let $\mathcal{C}$ be a finite collection of cells in $M^n$ and $U \subset M^n$. $\mathcal{C}$ is a *cell decomposition of* $U$ if $\mathcal{C}$ is a partition of $U$ and, if $n \ge 2$, the set $\prod_{n-1}(\mathcal{C}) := \{\prod_{n-1}(C) \mid C \in \mathcal{C}\}$ is a cell decomposition of $\prod_{n-1}(U)$

(2) If $Z \subset U$, then $\mathcal{U}$ is *compatible* with $Z$ if for every cell $C \in \mathcal{C}$ either $C \subseteq Z$ or $C \cap Z = \varnothing$

(3) If $\mathcal{C}$ and $\mathcal{D}$ are cell decompositions of $U$, we say that $\mathcal{D}$ is a *refinement* of $\mathcal{C}$, if $\mathcal{D}$ is compatible with every $C \in \mathcal{C}$

**Example 1.1.** Consider the set $S = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\right\}$ (13 cells for all of $\mathbb{R}^2$)

**Remark 1.6.**
- Let $\mathcal{C}$ be a cell decomposition of $U \subset M^{n+m}$ and let $x \in M^n$. Then $\mathcal{C}_x = \{C_x \mid C \in \mathcal{C}\}$ is a cell decomposition of $U_x = \{y \in M^m \mid (x,y) \in U\}$

  *Proof.* by induction on $m$ □

- Let $Z - 1, ..., Z_k \subset M^n$ and let $\mathcal{B}$ be the boolean algebra generated by them. Then: a cell decomposition of $M^n$ is compatible with the $Z_i$ iff it is compatible with all atoms of $\mathcal{B}$

  *Proof.* Every atom $B$ of $\mathcal{B}$ has the form $B = B_1 \cap ... \cap B_k$, where each $B_i$ is either $Z_i$ or $M^n \setminus Z_i$
  "$\Leftarrow$" clear.
  "$\Rightarrow$" If $\mathcal{C}$ is compatible with the $Z_i$, for each atom $B$ of $\mathcal{B}$

  - if $C$ contains every $B_i$, then $C$ is contained in $B$
  - If $C$ is disjoint for some $B_i$, then $C$ is disjoint from $\mathcal{B}$

  □

**Theorem 1.3** (Cell Decomposition Theorem). *(Siehe "Zellzerlegung O-minimalen Strukturen")*

$(I)_n$ *Let $S_1, ..., S_k \subset M^n$ be definable. Then there is a cell decomposition of $M^n$ that is compatible with every $S_i$*

$(II)_n$ *Let $f : S \to M$ be definable with $S \subset M^n$ definable. Then there is a cell decomposition $\mathcal{C}$ of $M^n$ compatible with $S$ such that for every $C \in \mathcal{C}$, $f \restriction_C$ is continuous.*

*Proof.* By induction

$n = 1$ $I_n$ follows easily from the definition of o-minimality, $II_n$ is the monotonicity theorem.

$n > 1$

**Lemma 1.7.** *Let $S \subset M^n$ be definable. The following are equivalent*

*(1) $S$ is* sparse *if (Definition)* $\mathsf{int}(S) = \emptyset$ *(Remark: A cell is sparse iff it is not open)*

*(2) The set $S' := \{x \in M^{n-1} \mid S_x \text{ is infinite}\}$ is sparse*

*(3) $S$ is* nowhere dense, *if (Definition)* $\mathsf{int}(\overline{S}) = \emptyset$

In particular, a finite union of sparse subsets of $M^n$ is sparse

*Proof.* 1 $\Rightarrow$ 2 Assume $S'$ is not sparse. Then we can find an open box $U \subset S'$. For any $x \in U$, since $S_x \subset M$ is infinite, $S_x$ contains an interval.
Fix a decomposition of $S_x$ as a union of finitely many open intervals and points. Let $I_x$ :=the first open interval in the decomposition.

$$i_x := \begin{cases} \inf I_x & \text{if } \inf I_x \in M \\ \text{a point in } I_x & \text{otherwise} \end{cases}$$

$$S_x := \begin{cases} \sup I_x & \text{if } \sup I_x \in M \\ \text{a point in } I_x \text{ greater than } i_x & \text{otherwise} \end{cases}$$

$i_x$ and $S_x$ are definable functions on $U$. By $II_{n-1}$, there is a cell decomposition $\mathcal{C}$ of $M^{n-1}$ with $U$ such that $i_x \mid_C$ and $S_x \mid_C$ are continuous for any $C \in \mathcal{C}$. Then there is an open cell $C' \in \mathcal{C}$ contained in $U$ (whence $(i_x \mid_{C'}, S_x \mid_{C'})_{C'}$ is an open cell and is contained in $S$)

(Note: $U$ cannot be a finite union of non-open cells)

□

BLabla, siehe Bachelorarbeit/Ziegler □

## 1.4 The Pila-Wilkie Theorem

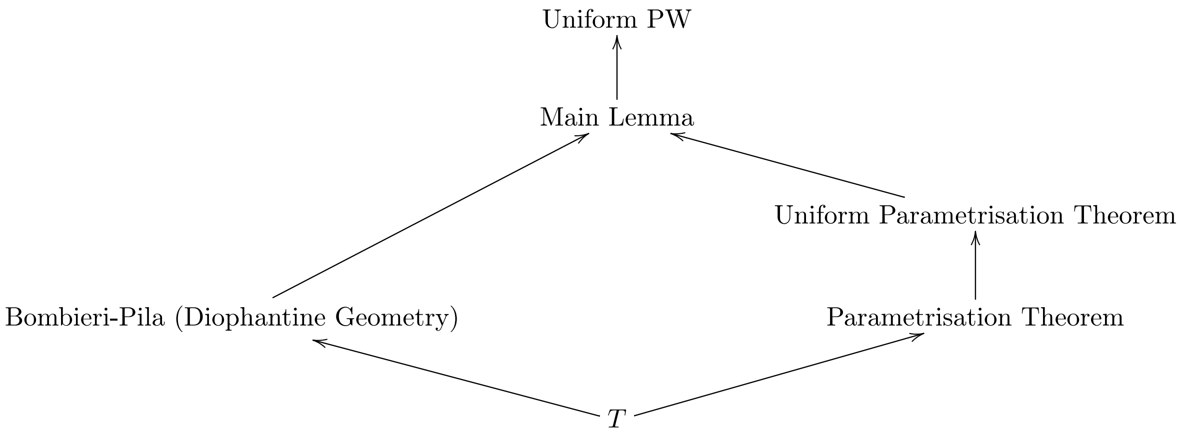
**Theorem 1.4.** *Let $\mathcal{R} = (\mathbb{R}, <, +, \cdot, -, 0, 1, ...)$ be an o-minimal expansion of the real field. Let $X \subset \mathbb{R}^n$ be definable in $\mathcal{R}$. Then for every $\varepsilon > 0$, there is $c = c(X, \varepsilon) > 0$ such that $\forall T \geq 1\ N(X^{tr}, T) \leq cT^{\varepsilon}$, where*

- *$X^{tr} = X \setminus X^{alg}$, $X^{alg} = \bigcup \{Y \mid Y \subset X$ is infitine, connected, semialgebraic$\}$*
- *For $X \subset \mathbb{R}^n$, $N(X, T) = |X(\mathbb{Q}, T)| := |\{x \in X \mid x \in \mathbb{Q}^n \wedge H(x) \leq T\}|$*

*For $q \in \mathbb{Q}^n, H(q) = \max(q_i)$, for $q \in \mathbb{Q}^{\times}, H(q) := \max(|a|, |b|)$, if $q = \frac{a}{b}, gcd(a, b) = 1$, $H(0) = 0$.*

**Theorem 1.5** ((Uniform Pila-Wilkie))**.** *Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +, \cdot, -, 0, 1)$. Let $(X_b)_b$ be a definable family of subsets of $\mathbb{R}^n$. For every $\varepsilon > 0$, there exists a family $(Y_b)_b$ and $c = c((X_b)_b, \varepsilon)$ such that for every $b \in B$*

- *$Y_b \subset X_b^{alg}$*
- *$\forall T \geq 1, N(X_b \setminus Y_b, T) \leq cT^{\varepsilon}$*




**Definition 1.11.** A *hypersurface of degree $d$* in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n \mid f(x) = 0\}$ where $f \in \mathbb{R}[X]$ has degree $d$.

**Definition 1.12.** Let $X \subset \mathbb{R}^n, k \in \mathbb{Z}_+$. A *partial $k$-parametrisation* of $X$ is a function $f : (0,1)^{\mathsf{dim} X} \to X$ such that $\forall \alpha \in \mathbb{N}^{\mathsf{dim} X}$ with $|\alpha| \leq k$ ($|\alpha| := \sum \alpha_i$):

- $f^{(\alpha)}$ is continuous
- $\underbrace{\left|f^{(\alpha)}\right|}_{|y| := \max |y_i|} \leq 1, \forall x \in (0,1)$

**Definition 1.13.** A *$k$-parametrisation* of $X$ is a finite set $S$ of partial parametrisations of $X$ such that $\bigcup_{f \in S} \mathsf{im}(f) = X$

**Theorem 1.6** (Bombieri-Pila)**.** *Given $0 < m < n, d > 0$, there exist $k = k(m,n,d) \in \mathbb{Z}_+, \varepsilon = \varepsilon(m,n,d) > 0$ and $c = c(m,n,d) > 0$ such that if $f : (0,1)^m \to X$ is a $k$-parametrisation of its image, then $X(\mathbb{Q},T) \subset$union of $\leq cT^\varepsilon$ hypersurfaces of degree $d$. Moreover, $\varepsilon \to 0$ as $d \to \infty$.*

Let $\mathcal{M}$ be an o-minimal expansion of a real closed field $M$.

**Definition 1.14.**

- An element $a \in M$ is *strongly bounded* if there is $N \in \mathbb{N}$ such that $|a| \leq N$
- $a \in M^n$ is strongly bounded if there is $N \in \mathbb{N}$ such that $|a| := \max |a_i| \leq N$
- A subset $A$ of $M^n$ is strongly bounded if there is $N \in \mathbb{N}$ such that $|a| \leq N \forall a \in A$

**Theorem 1.7** (Parametrisation Theorem)**.** *Let $X$ be a strongly bounded definable subset of $M^n$, for every $k \in \mathbb{Z}_+$, there exists a definable $k$-parametrisation of $X$.*

**Corollary 1.4.** *Let $m, r \geq 1, X \subset (0,1)^m$ definable. Then there exist a finite set $S$ of functions $(0,1)^{\mathit{dim}X} \to X$ of class $C^r$ such that $\bigcup_{\phi \in S} \mathit{im}(\phi) = X$ and $\left|\phi^{(\alpha)}(x)\right| \leq 1$ for all $\phi \in S, \alpha \in \mathbb{N}^{\mathit{dim}X}$ with $|\alpha| \leq r$ and $x \in (0,1)^{\mathit{dim}X}$.*

**Definition 1.15.** A partial $r$-parametrisation of $X$ is a $C^r$-function $f : (0,1)^{\mathsf{dim}X} \to X$ such that $\forall \alpha \in \mathbb{N}^{\mathsf{dim}X}$ with $|\alpha| \leq r, \forall x \left|f^{(\alpha)}(x)\right| \leq N$ for some $N \in \mathbb{N}$.

*proof of corollary.* Let $S$ be an $r$-parametrisation of $X$. Cover $(0,1)^{\mathsf{dim}X}$ with $N^{\mathsf{dim}X}$ cubes of side $\frac{1}{N}$ and for each cube $K$, let $\lambda_K : (0,1)^{\mathsf{dim}X} \to K$ be the obvious bijection. Let $S := \left\{\phi \circ \lambda_K \mid \phi \in S^K, K \text{ is one of the cubes}\right\}$, $d := \mathsf{dim}X$.

$\mathbb{R}^\phi \xrightarrow{\lambda} \mathbb{R}^d \xrightarrow{\phi} \mathbb{R}^n$.

$$\frac{\partial(\phi \circ \lambda)}{\partial x_i}(x) = \frac{\partial \phi}{\partial \lambda}(\lambda(x)) \cdot \frac{\partial \lambda}{\partial x_i}(x) = \frac{1}{N} \cdot \frac{\partial \phi}{\partial x_i}(\lambda(x))$$

$$\frac{\partial(\phi \circ \lambda)}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_d} x_d} = \left(\frac{1}{N}\right)^{|\alpha|} \frac{\partial \phi}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_d} x_d} \leq \frac{1}{N} \cdot N$$

□

**Corollary 1.5** (Uniform parametrisation Theorem)**.** *Let $n, m, r \geq 1$, $X \subset (0,1)^m \times M^n$ definable. Then there exist $N \in \mathbb{N}$ and for all $y \in M^n$ a finite set $S_y$ of $N$ $C^r$-functions $(0,1)^{\mathit{dim}X_y} \to X_y$ such that (1) $\bigcup_{\phi \in S_y} \mathit{im}(\phi) = X_y$ and (2) $\left|\phi^{(\alpha)}(x)\right| \leq 1$ for all $y, \phi \in S_y, \alpha \in \mathbb{N}^{\mathit{dim}X_y}$ with $|\alpha| \leq r$ and $x \in (0,1)^{\mathit{dim}X_y}$.*

*Proof.* Suppose not, i.e. for every $N \in \mathbb{N}$ there is $y_N \in M^n$ such that ... but this implies that there is $y \in M^n$ such that "Parametrisation theorem does not hold for $X_y$".

For $N \in \mathbb{Z}_+$, let $\Gamma_N(v)$ be the set of formulas expressing "for every set of $N$ functions satisfying (2), the union of their images is not $X_v$". Let $\Gamma(v) = \bigcup_N \Gamma_N(v)$.

$\Rightarrow \Gamma(v)$ is finitely satisfiable. Compactness: there is $\mathcal{N} > M$ and $y \in \mathcal{N}$ such that $\mathcal{N} \vDash \Gamma(y)$. □

Main Lemma - siehe Skript.

**Theorem 1.8** (Laurent '80s, conjectured by Lang for curves, "Manin Mumford Conjecture for $\mathbb{C}_m$" (multiplicative group). *Suppose $V \subset (\mathbb{C}^\times)^d$ is an irreducible subvariety. Then there are finitely many algebraic subgroups $B_1, ..., B_n$ of $(\mathbb{C}^\times)^d$ and $b_1, ..., b_n \in (\mathbb{C}^\times)^d$ such that*

- $b_i B_i \subset V$ *for all* $i$
- $V \cap \mu^d = \bigcup_i b_i(B_i \cap \mu^d)$ *(where* $\mu^d = \mathit{Tor}((\mathbb{C}^\times)^d)$ *and* $(B_i \cap \mu^d) = \mathit{Tor}(B_i)$ *and* $\mu$ :=*roots of unity*$\subset \mathbb{C}^\times$*)*

*Fact:* Every algebraic subgroup of $(\mathbb{C}^\times)^d$ is defined by a finite set of equations of the form:

$$y_1^{m_{11}}...y_n^{m_{1n}} = 1, m_i \in \mathbb{Z}$$

$$...$$

$$y_1^{m_{k1}}...y_n^{m_{kn}} = 1$$

and $\dim B = d - rk_{\mathbb{Q}}(m_{ij})$. $LB = \{x \mid M \cdot x = 0\} \subset \mathbb{C}^d$. $\dim LB = \dim B$. In particular,

- If $V$ contains no cosets of infinite algebraic subgroups, then $V \cap \mu^d$ is finite

*Proof.* See "The case of Tori" onwards. $(\mathbb{C}^\times)^d = \mathbb{G}_m^d$ □