Vorlesungsmitschrift Descriptive Set Theory

Aktueller Stand: 26. Juni 2012

Overview

We Start with a polish space X (e.g. $X = \mathbb{R}$) trivial: $X, \emptyset$ Simple Sets: Open sets:$\underset{\sim}{\sum}{}^0_1(X)$
Closed sets:$\underset{\sim}{\prod}{}^0_1(X)$ goal: establishing a hierarchy of complexity (via ordinal numbers $\Rightarrow$
Borel sets ($\underset{\sim}{\Delta}{}^1_1(X)$) are class ω_1)
α countable ordinal, Difference hierarchy: α countable ordinal and successors

1 Basic Definitions

Definition 1.1 $(X, \mathcal{T})$ is a *topological space* if $\mathcal{T}$ is a *topology* of X.
$\mathcal{T}$ is a topology, if:

- $\emptyset, X \in \mathcal{T}$
- $\mathcal{T}$ is closed under finite intersections
- $\mathcal{T}$ is closed under unions

$U \in \mathcal{T}$ are called open sets.

Example 1.1 $(\mathbb{R}, \mathcal{T})$: T is closure under unions and finite intersections of
$\{(a, b), (-\infty, A), (a, \infty) \mid a, b \in \mathbb{R}\}$
$(X, \mathcal{T})$ topological space, $Y \subseteq X$, the *relative topology* on Y is $\{U \cap Y \mid U \in \mathcal{T}\}$

$(X, \mathcal{T})$ is topological space, $F_\sigma = \{\bigcup_n F_n \mid X \setminus F_n \in \mathcal{T}\}$
$G_\delta = \{\bigcap U_n \mid U_n \in \mathcal{T}\}$

Definition 1.2 $B \subseteq \mathcal{P}(X)$ is a *basis* for $(X, \mathcal{T})$ if for every $U \in \mathcal{T} : U = \bigcup_\alpha A_\alpha$ for some $A_\alpha \in B$

B is a *subbasis* if the collection of all finite intersections of sets from B is a basis

Definition 1.3 $(X, \mathcal{T})$ is *second-countable* if it admits a countable basis
Example: $(\mathbb{R}, \mathcal{T})$, $B = \{(p,q), (-\infty, p), (p, \infty) \mid p, q \in \mathbb{R}\}$

Definition 1.4 $f : X \to Y$ is *continuous* if $f^{-1}(U)$ is open (in X) for every open $U \subset Y$
$\Rightarrow$ For every $x_n \to x$: $f(x_n) \to f(x)$
$x_n \to x$ if for every open U such that $x \in U$: $\exists N \forall m \geq N : x_m \in U$

Definition 1.5 (X, d) is a *metric space* if $d : X \times X \to \mathbb{R}^+$ such that

- $d(x,y) = 0 \Leftrightarrow x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$

Example: $(\mathbb{R}, d)$, $d(r, r') = |r - r'|$

$B_\varepsilon(x) := \{y \in X \mid d(x,y) < \epsilon\}$

d generates $\mathcal{T}_d$, where $\mathcal{T}_d$ is the topology generated by $\{B_\varepsilon(x) \mid x \in X, 0 < \varepsilon \in \mathbb{R}^+\}$

$(X, \mathcal{T})$ is called *metrizable* if there is a metric d on X such that $\mathcal{T} = \mathcal{T}_d$

Definition 1.6 $(X, \mathcal{T})$, $D \subseteq X$ is called *dense*, if $\forall U \in \mathcal{T} : U \cap D \neq \emptyset$
Example: $\mathbb{Q}$ is dense in $\mathbb{R}$

$(X, \mathcal{T}$ is *separable* if there is a dense and countable $D \subseteq X$

Proposition : If $(x, \mathcal{T})$ is metrizable, then $(X, \mathcal{T}$ is separable iff it is second-countable

$(X_i, \mathcal{T}_i), i < I, I \leq \omega$ ("at most countable"). $(\prod_i X_i, \mathcal{T})$, $\mathcal{T}$ is generated by $\left\{\pi_i^{-1}(U) \mid i < I, U \in \mathcal{T}_i\right\}$
(for finite equivalent to "box topology") $\pi_i : \prod X_i \to X_i$
$(x_0, x_1, ..., x_i, ...) \mapsto x_i$
$(X_i, d_i), i < I, I \leq \omega$, define the metric d on $\prod_i X_i$ by $d(\vec{x}, \vec{y}) = \sum\limits_{n=0}^{\infty} \frac{1}{2^{n+2}} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n))}$
$\mathcal{T}_d = \mathcal{T}$

Definition 1.7 (X, d) metric space, (x_n) is *cauchy*, if $\forall \varepsilon \in \mathbb{R} \setminus \{0\} \exists N \forall m, m' \geq N : d(x_m, x_{m'}) < \varepsilon$

(X,d) is *complete* if every Cauchy-Sequence converges to some point of X
Example: (X,d) is complete

Definition 1.8 $(X,\mathcal{T})$ is *polish*, if it is second-countable and completely metrizable (i.e. there is a complete metric d, such that $\mathcal{T}=\mathcal{T}_d$)
$\Rightarrow$ separable
(X,d) is a polish metric space, if $(X,\mathcal{T}_d)$ is polish
Examples:

- $(\mathbb{R},\mathcal{T})$ is polish
- $\mathbb{R}^n,\mathbb{C}^n$ are polish
- $((0,1),\mathcal{T})$ is polish (homeomorphic to the real line), but with a different metric
- $([0,1],\mathcal{T})$ is polish with usual metric
- Let X be countable and discrete, $\mathcal{T}=\mathcal{P}(X)$ with Basis $B=\{\{x\}\mid x\in X\}$ and metric $d(x,y)=\begin{cases}0 & \text{if } x=y\\ 1 & \text{if } x\neq y\end{cases}$
  $\mathcal{T}$ is complete

Take $(\omega,\mathcal{T})$[1], $\mathcal{T}$ discrete (therefore polish).
For $i\in\omega$, let $(X_i,\mathcal{T}_i)=(\omega,\mathcal{T})$. The Baire space is $(\prod_i X_i,\mathcal{T})$.

Theorem 1.1 if $(X_i,\mathcal{T}_i)$ is polish, then $(\prod_i X_i,\mathcal{T}_i)$ is polish.
${}^\omega\omega := \mathcal{N} = \mathbb{N}^\omega$, the sets $\{x\in {}^\omega\omega \mid x_i=n\}$ for $i\leq\omega, n\in\mathbb{N}$ form a subbasis for the product topology (countable).
Let $s=(s_0,s_1,...,s_n)$ be a finite sequence of natural numbers, $N_s=\{x\in {}^\omega\omega \mid s\subseteq x, x\uparrow n+1=s\}$ $\forall i\leq n: x(i)=s(i)$

Exercise: $\{N_g \mid \text{for } s \text{ finite sequences of natural numbers}\}$ is a basis for the topology of ${}^\omega\omega$

Take $(\{0,1\},\mathcal{T})$ discrete. The countable product is the Cantor space ${}^\omega 2$

Fact: $1\omega\omega$ is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$
${}^\omega 2$ is homeomorphic to the *cantor ternary set* $(\subseteq[0,1])$

If X is a separable Banach space with norm $||\cdot||$ then letting $d(x,y)=||x-y||$ we get the polish metric space (X,d)

[1] In this case $\omega=\mathbb{N}$

Example 1.2

- $\mathfrak{L}^p, c_0$ are polish, $C^0([0,1])$ with the sup norm $||f|| = \sup\{|f(x)| \mid x \in [0,1]\}$ is polish metric space
- Let X be a polish space, $K(X) = \{K \subset X \mid K \text{ compact}\}$ endowed with the Vietoris topology = the topology generated by sets of the form $\{K \in K(X) \mid K \subset U\}$ or $\{K \in K(X) \mid K \cap U \neq \emptyset\}$ for $U \subset X$ open is a polish space
- If $X_n, n \in \omega$ are polish spaces, then $\prod_n X_n$ endowed with the product topology is polish
- Let X be polish. Then $A \subset X$ is polish iff A is G_δ

2 Trees

Definition 2.1 $A \neq \emptyset,\ n \in \omega$
A Sequence s on A is an object of the form $\langle s_0, s_1, ..., s_{n-1}\rangle$ where each $s_i \in A$. The number n is called *length* of s and is denoted by lh(s)
${}^nA = \{\text{sequences of length } n \text{ on } A\}$, ${}^{<\omega}A = \bigcup\limits_{n\in\omega} {}^nA$
$s \in {}^nA$ and $m \leq n$ then $s \upharpoonright m = \langle s_0, ..., s_{m-1}$ is the *restriction* of s to m
$s, t \in {}^{<\omega}A,$ s and t are *compatible* if either $s = t \upharpoonright \text{lh}(s)$ or $t = s \upharpoonright \text{lh}(t)$
The *concotenation* of $s, t \in {}^{<\omega}A$ is $s \frown t = \langle s_0, ..., s_{\text{lh}(s)-1}, t_0, ..., t_{\text{lh}(t)-1}\rangle$
(if $s \in {}^nA$ and $a \in A$, we write $s \frown a$ for $s \frown \langle a\rangle$)
An ω-sequence is $\langle x_0, x_1, ...\rangle$ with $x_i \in A \forall i \in \omega$
${}^\omega A = \{\text{all } \omega\text{-sequences on } A\}$
$s \in {}^{<\omega}A, x \in {}^\omega A$, then $s \frown x = \langle s_0, ..., s_{\text{lh}(s)-1}, x_0, x_1, ...\rangle$

Definition 2.2 A *tree* on A is a subset $T \subset {}^{<\omega}A$ closed under initial segments, e.g. if $t \in T$and $n <$lh(t) then $t \upharpoonright n \in T$ The elements of T are called *notes*

Definition 2.3 $T \subset {}^{<\omega}A$ tree, an *infinite branch* of T is $x \in {}^\omega A$ such that
$x \upharpoonright n \in T \forall n \in \omega$
The *body* of T is the set $[T] := \{x \in {}^\omega A \mid x \text{ is an infinite branch of } T\}$

Definition 2.4 T is called *pruned* if $\forall t \in T \exists a \in A : t \frown a \in T$ ($\Leftrightarrow T$ has no *terminal* node, where $t \in T$ is terminal if $t \frown a \notin T$ for $a \in A$)

Definition 2.5 T is *well-founded* if $[T] = \emptyset$

${}^{\omega}2 = {}^{\omega}\{0,1\} <-$ Cantor space

Theorem 2.1 The product topology on ${}^{\omega}A$ is generated by $N_s = \{x \in {}^{\omega}A \mid x \restriction \mathrm{lh}(s) = s\}$ for $s \in {}^{<\omega}A$

A compatible metric on ${}^{\omega}A$ is given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2^{n+1}} & \text{where } n \text{ is least s.t. } x \restriction n = y \restriction n \text{ and } x_n \neq y_n \end{cases}$$

$s \in {}^{<\omega}A \Rightarrow N_s = B(x, \frac{1}{2^{\mathrm{lh}(s)}})$ for $x \in N_s$

Proof: if $y \in B(x, \frac{1}{2^{\mathrm{lh}(s)}})$, then $d(x,y) \leq \frac{1}{2^{\mathrm{lh}(s)+1}}$
$\Leftrightarrow x \restriction \mathrm{lh}(s) = y \restriction \mathrm{lh}(y) \Leftrightarrow y \in N_s$

□

The metric is complete:

Proof: Let $< x_n \mid n \in \omega$ be a cauchy-sequence in ${}^{\omega}A$. For $n \in \omega \exists N \in \omega$: $\forall m, m' \geq N d(x_m, x_{m'}) < \frac{1}{2^{n+1}}$
$\Rightarrow d(x_m, x_{m'}) \leq \frac{1}{2^{n+2}} \Rightarrow x_m \restriction n+1 = x_{m'} \restriction n+1 = s_n$

□

Exercises $x_n \to x$ where $x \restriction n+1 = s_n$
$N_s \cap N_t = \emptyset \Leftrightarrow s,t$ are incompatible
$s \subset t (\Leftrightarrow t \restriction \mathrm{lh}(s) = s) \Rightarrow N_t \subset N_s$

Theorem 2.2 The map $T \to [T]$ for a pruned tree T on A is a bijection between pruned trees and closed subsets of ${}^{\omega}A$. The inverse is $F \to T_F = \{x \restriction n \mid x \in F, n \in \omega\}$

Proof sketch: Z.z. $[T]$ is closed: Take $x \notin [T] \Rightarrow \exists n \in \omega : x \restriction n \notin T$. Then $N_{x \restriction n} \cap [T] = \emptyset$
T_F is a pruned tree. If $t \in T_F$ and $M \leq \mathrm{lh}(t)$. $\Rightarrow \exists x \in F, n \in \omega : t = x \restriction n \Rightarrow x \restriction m \in T_F$, so T_F is a tree. Let $t \in T_F$: then $\exists x \in F, n \in \omega : t = x \restriction n$. Then $x \restriction n+1 > t$ and $x \restriction n \neq s \in T_F$

□

Definition 2.6 Let S, T be trees on A and B resp. A map $\varphi : S \to T$ is *monotone*, if $s \subset t \Rightarrow \varphi(s) \subset \varphi(t)$
$D(\varphi) = \left\{ x \in [S] \mid \lim_{n \to \infty} \mathrm{lh}(\varphi(x \restriction n)) = \infty \right\}$

$$\varphi^* : D(\varphi) \to [T]$$
$$x \to \bigcup_{n\in\omega} \varphi(x \upharpoonright n)$$

Theorem 2.3 The set $D(\varphi)$ is G_δ and φ^* is continuous. Conversely, if $f : G \to [T]$, where $G \subset [S]$ is G_δ is continuous, then there is a monotone $\varphi : S \to T$ such that $\varphi^* = f$

Proof: We have $x \in D(\varphi) \Leftrightarrow \forall n \exists m$:lh$(\varphi(x \upharpoonright)) \geq n$. Therefore $D(\varphi) = \bigcap_n U_n$, where $U_n = \bigcup_m \underbrace{\{x \mid \text{lh}(\varphi(x \upharpoonright)) \geq n\}}_{A_{n,m}}$=open, because $N_{x\upharpoonright n} \subset A_{n,m}$
$\Rightarrow D(\varphi)$ is G_δ.
To see that φ^* is continuous, notice that $V_t = N_t \cap [T]$ (for $t \in {}^{<\omega}B$) form a basis for the topology of $[T]$
$\mathcal{Z}(\varphi^*)^{-1}(V_t)$ is open: $= \bigcup \{N_s \cap [S] \mid \varphi(s) \supset t, s \in S\}$

□

$\varphi : S \to T$ monotone, $s \leq t \Rightarrow \varphi(s) \leq \varphi(t)$
$D(\varphi) = \left\{\mathcal{Z} \in [S] \mid \lim_{n\to\infty} lh(\varphi(s)) = \infty\right\}$ $\varphi^* : D(\varphi) \to [T] : x \to \bigcup_n \varphi(x \upharpoonright n)$

Theorem 2.4

1. $D(\varphi)$ is G_δ and φ^* is continuous
2. If $f : G \to [T]$ is continuous with $G \subset [S]$ is G_δ, then $\exists \varphi : S \to T$ monotone such that $f = \varphi^*$

Exercise let $\varphi : S \to T$ monotone. φ is called *Lipschitz* with $L < 1$ if $\forall s \in S lh(\varphi(s)) = lh(s)$. Show that $d(\varphi^*(x), \varphi^*(y)) \leq d(x, y) \forall x, y \in D(\varphi)$

Definition 2.7 X, Y topological spaces, $f : X \to Y$ is *topological embedding* if f is continuous, injective, f^{-1} :range$(f) \to Y$ is continuous as well.

Definition 2.8

- $x \in X$ is limit if every open set containing x contains at least one other point
- X is called *perfect* if $\forall x \in X x$ is limit
- $P \subset X$ is perfect if P is closed and perfect

Definition 2.9 QCantor Scheme on X is a family $(A_s), s \in {}^{<\omega}2$ as of X such that

1. $A_{s\frown 0} \cap A_{s\frown 1} = \emptyset$ from $s \in {}^{<\omega}2$
2. $A_{s\frown i} \subset A_s$

Theorem 2.5 Let X be a nonempty perfect polish space, then there si an embedding of ${}^{\omega}2$ in X

Theorem 2.6 Let X be a polish space. Then $X = P \cap C$ with $P \cap C \neq \emptyset$, P is perfect in X and C is countable and open

Definition 2.10 Let $(X, \mathcal{T})$ be a topological space. The class $B(X) = B(X, \mathcal{T})$ of Borel sets of X is the minimal σ-Algebra containing $\mathcal{T}$, i.e. the smallest class C of subsets of X such that

1. C is closed under complements
2. C is closed under countable unions
3. $C \supset \mathcal{T}$

Theorem 2.7 For every X, $B(X)$ is the smallest collection $S \subset \mathcal{P}(X)$ such that:

i S is closed under countable unions and countable intersections

ii S contains all open and closed sets of X

Proof: Claim: $S \subset B(X)$:
$B(X)$ satisfies (ii), because $\mathcal{T} \subset B(X)$ and $B(X)$ closed under complements
$\mathbb{Z}$: closed under countable intersections: Let $\{B_n \mid n \in \omega\} \subset B(X)$, then $\bigcap_{n\in\omega} B_n = X \setminus \underbrace{(\bigcup_{n\in\omega} \underbrace{(X \setminus B_n)}_{\in B(X)})}_{\in B(X)} \in B(X)$

Claim: $B(X) \subset S$:
Let $S' = \{A \subset X \mid A, x \setminus A \in S\} \subset S$. Then S' satisfies (1).
S' satisfies (3) by (ii). Finally, S' satisfies (2): $\{B_n \mid n \in \omega\} \subset S'$, then $\bigcup_{n\in\omega} B_n \in S$ (by (i)) and $X \setminus \bigcup_{n\in\omega} B_n = \bigcap_{n\in\omega} \underbrace{(X \setminus B_n)}_{\in S} \in S$ (by (i)).

Hence $\bigcup_{n\in\omega} B_n \in S'$

□

Let X be metrizable. $\underset{\sim}{\Sigma}_1{}^0(X) := \mathcal{T}$

$\underset{\sim}{\Pi}_\xi{}^0(X) = \left\{ X \setminus A \mid A \in \underset{\sim}{\Sigma}_\xi{}^0(X) \right\}$

$\underset{\sim}{\Sigma}_\xi{}^0(X) = \left\{ \bigcup_{n\in\omega} A_n \mid A_n \in \bigcup_{\beta<\xi} \underset{\sim}{\Pi}_\beta{}^0(X) \right\}$ for $\xi < \omega_1$ (countable ordinal)

So:

$\underset{\sim}{\Sigma}_1{}^0$ open sets of X

$\underset{\sim}{\Pi}_1{}^0$ closed sets of X

$\underset{\sim}{\Sigma}_2{}^0$ countable unions of closed sets $= F_\sigma$

$\underset{\sim}{\Pi}_2{}^0$ countable intersections of open sets $= G_\delta$

$\vdots$

$\underset{\sim}{\Delta}_\xi{}^0 = \underset{\sim}{\Sigma}_\xi{}^0 \cap \underset{\sim}{\Pi}_\xi{}^0(X) = \left\{ A \subset X \mid A, X \setminus A \in \underset{\sim}{\Sigma}_\xi{}^0 \right\}(X)$

$\underset{\sim}{\Delta}_\xi{}^0(X) \subset \underset{\sim}{\Sigma}_\xi{}^0(X), \underset{\sim}{\Pi}_\xi{}^0(X)$

Lemma 2.1 X metrizable, second-countable. For every $\xi < \omega_1, \underset{\sim}{\Sigma}_\xi{}^0(X) : \underset{\sim}{\Pi}_\xi{}^0(X) \subset \underset{\sim}{\Delta}_{\xi+1}{}^0(X)$

Proof: By induction on ξ:

$\xi = 1$: Claim: $\underset{\sim}{\Sigma}_1{}^0(X) \subset \underset{\sim}{\Delta}_2{}^0(X) = \underset{\sim}{\Sigma}_2{}^0(X) \cap \underset{\sim}{\Pi}_2{}^0(X)$

Let $U \in \underset{\sim}{\Sigma}_1{}^0(X)$. Then setting $U_n = U$ for every $n \in \omega, U = \bigcap_{n\in\omega} U_n \in \underset{\sim}{\Pi}_2{}^0(X)$

Let d be a compatible metric, and $D \subset X$ be countable and dense. Then $\{B(x,q) \mid x \in D, q \in \mathbb{Q}\}$ is a countable basis for the topology of X. Therefore, since $\underset{\sim}{\Sigma}_2{}^0(X)$ is closed under countable unions, it is enough to prove that $B(x,q) \in \underset{\sim}{\Sigma}_2{}^0(X)$.

Let q_n countable increasing sequence of rational numbers such that $q_n < q$ and $q_n \to q$. Then $B(x,q) = \bigcup_{n\in\omega} \underbrace{cl(B(x,q_n))}_{\{y\in X \mid d(x,y)\leq q_n\}}$ [2]

$\xi > 1$ and $A \in \underset{\sim}{\Sigma}_\xi{}^0(X)$: Taking $A_n = A$ for all $n \in \omega$, $A = \bigcap_{n\in\omega} A_n = X \setminus \bigcup_{n\in\omega} \underbrace{(X \setminus A_n)}_{\in \underset{\sim}{\Pi}_\xi{}^0(X)} \in \underset{\sim}{\Pi}_{\xi+2}{}^0(X)$

$A \in \underset{\sim}{\Sigma}_\xi{}^0(X) \Rightarrow A = \bigcup_{n\in\omega} A_n$ with $A_n \in \bigcup_{\beta<\xi} \underset{\sim}{\Pi}_\beta{}^0(X) \in \underset{\sim}{\Delta}_{\beta+1}{}^0$

[2] cl=closure

$\Rightarrow$ by induction hypothesis $A \in \underset{\sim}{\Sigma}_{\xi+1}{}^0(X)$

$\square$

Lemma 2.2 $B(X) = \bigcup_{\xi<\omega_2} \underset{\sim}{\Sigma}_\xi{}^0(X) = \bigcup_{\xi<\omega_2} \underset{\sim}{\Pi}_1{}^0(X) = \bigcup_{\xi<\omega_2} \underset{\sim}{\Delta}_\xi{}^0(X)$.

$$\underset{\sim}{\Delta}_\xi{}^0(X) \subset \underset{\sim}{\Sigma}_\xi{}^0(X), \underset{\sim}{\Pi}_\xi{}^0(X) \subset \underset{\sim}{\Delta}_{\xi+1}{}^0$$

Property If $X \subset Y$, then
$\underset{\sim}{\Sigma}_\xi{}^0(X) = \underset{\sim}{\Sigma}_\xi{}^0(Y) \mid X = \left\{A \cap X \mid A \in \underset{\sim}{\Sigma}_\xi{}^0(Y)\right\}$
$\underset{\sim}{\Pi}_\xi{}^0(X) = \underset{\sim}{\Pi}_\xi{}^0(Y) \mid X = \left\{A \cap X \mid A \in \underset{\sim}{\Pi}_\xi{}^0(Y)\right\}$
(Proof by induction on $\xi < \omega_1$)

Theorem 2.8 For each X metrizable and second-countable, for every $\xi < \omega_1$ the classes $\underset{\sim}{\Sigma}_\xi{}^0(X), \underset{\sim}{\Pi}_\xi{}^0(X)$ and $\underset{\sim}{\Delta}_\xi{}^0(X)$ are closed under finite intersections, finite unions and continuous preimages, (i.e. $\forall A \in \underset{\sim}{\Sigma}_\xi{}^0(X)$ and $f : X \to X$ continuous, $f^{-1}(A) \in \underset{\sim}{\Sigma}_\xi{}^0(X)$, respectively for Π, Δ)
Moreover, $\underset{\sim}{\Sigma}_\xi{}^0(X)$ is closed under countable unions, $\underset{\sim}{\Pi}_\xi{}^0(X)$ is closed under countable intersections, $\underset{\sim}{\Delta}_\xi{}^0(X)$ is closed under complements.
Finally, if $A \in \underset{\sim}{\Sigma}_\xi{}^0(X)$ and $f : Y \to X$ is continuous, then $f^{-1}(A) \in \underset{\sim}{\Sigma}_\xi{}^0(Y)$ (respectively for Π, Δ)

Definition 2.11 Let Γ be one of $\underset{\sim}{\Sigma}_\xi{}^0, \underset{\sim}{\Pi}_\xi{}^0, \underset{\sim}{\Delta}_\xi{}^0$. We denote by $\Gamma(X)$ the class $\underset{\sim}{\Sigma}_\xi{}^0(X)$ / $\underset{\sim}{\Pi}_\xi{}^0(X)$ / $\underset{\sim}{\Delta}_\xi{}^0(X)$
We say that $U \subset Y \times X$ is *Y-universal for* $\Gamma(X)$, if $U \subset \Gamma(Y \times X)$ and $\Gamma(X) = \{U_y \mid y \in Y\}$, where $U_y = \{x \in X \mid (y, x) \in U\}$ (*parametrization* or *coding* of $\Gamma(X)$, y is a *code* for U_y)

$\breve{\Gamma}(X) = \{X \setminus A \mid A \in \Gamma(X)\}$
$\Delta(X) = \Gamma(X) \cap \breve{\Gamma}(X)$

Theorem 2.9 Let X be a seperable metrizable space. Then for each $\xi < \omega_1$ there is a ${}^\omega 2$-*universal* set for $\underset{\sim}{\Sigma}_\xi{}^0(X)$ (resp. $\underset{\sim}{\Pi}_\xi{}^0(X)$)

Proof: by induction on $\xi < \omega_1$

□

Exercise Show that if U is Y-universal for $\Gamma(X)$, then $(Y \times X) \setminus U$ is Y-universal for $\breve{\Gamma}(X)$

$\Gamma(X)$ be one of $\underset{\sim}{\Sigma}_\xi{}^0(X), \underset{\sim}{\Pi}_\xi{}^0(X)$ over a separable metrizable X. We say $U \subset {}^\omega 2 \times X$ is ${}^\omega 2$-universal for $\Gamma(X)$ if $U \in \Gamma({}^\omega 2 \times X)$ and $\Gamma(X) = \{U_y \mid y \in {}^\omega 2\}$ (and $U_y = \{x \in X \mid (y,x) \in U\}$)

Theorem 2.10 X as above, then there is a ${}^\omega 2$-universal set for $\Gamma(X)$

Proof: By Induction:

- There is a ${}^\omega 2$-universal set for $\underset{\sim}{\Sigma}_1{}^0(X)$
- If U is ${}^\omega 2$-universal for $\Gamma(X)$ then ${}^\omega 2 \times X \setminus U$ is ${}^\omega 2$-universal for $\breve{\Gamma}(X) = \{X \setminus A \mid A \in \Gamma(X)\}$
- Assume that for every $\eta < \xi$ there is a ${}^\omega 2$-universal U_η from $\underset{\sim}{\Pi}_\eta{}^0(X)$. Choose $(\eta_n)_{n\in\omega}$ such that $\eta_n \leq \eta_{n+1}$ and $\sup(\eta_n + 1) = \xi$ (if $\xi = \eta + 1$...)

□

Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Given $y \in {}^\omega 2$ and $n \in \mathbb{N}$, let $(y)_n \in {}^\omega 2$ be defined by $(y)_n(m) = y(\langle n, m \rangle)$

Definition 2.12 Let $U \subset {}^\omega 2 \times X$ be defined by $(y,x) \in U \Leftrightarrow \exists n(((y)_n, x) \in U_{\eta_n})$

Theorem 2.11 Let X be an uncountable polish space. Then for each ξ, $\underset{\sim}{\Sigma}_\xi{}^0(X) \neq \underset{\sim}{\Pi}_\xi{}^0(X)$. Therefore $\underset{\sim}{\Delta}_\xi{}^0(X) \subsetneq \underset{\sim}{\Sigma}_\xi{}^0(X) \subsetneq \underset{\sim}{\Delta}_{\xi+1}{}^0(X)$

Proof: Since X is uncountable, then we can assume w.l.o.g. ${}^\omega 2 \subset X$
If $\underset{\sim}{\Sigma}_\xi{}^0(X) = \underset{\sim}{\Pi}_\xi{}^0(X)$, then $\underset{\sim}{\Sigma}_\xi{}^0({}^\omega 2) = \left\{A \cap {}^\omega 2 \mid A \in \underset{\sim}{\Sigma}_\xi{}^0(X)\right\} = \left\{A \cap {}^\omega 2 \mid A \in \underset{\sim}{\Pi}_\xi{}^0(X)\right\} = \underset{\sim}{\Pi}_\xi{}^0({}^\omega 2)$

Let $U \subset {}^\omega 2 \times {}^\omega 2$ be a ${}^\omega 2$-universal set for $\underset{\sim}{\Sigma_\xi}^0({}^\omega 2)$. Define $A \subset {}^\omega 2$ by setting $y \in A \Leftrightarrow (y,y) \notin U$. Since $x \mapsto (y,y)$ is continuous and $U \in \underset{\sim}{\Sigma_\xi}^0({}^\omega 2 \times {}^\omega 2)$, then $A \in \underset{\sim}{\Pi_\xi}^0(X)$. If $A \in \underset{\sim}{\Sigma_\xi}^0({}^\omega 2)$, then $A = U_{y_0}$ for some $y_0 \in {}^\omega 2$. Then $y_0 \in A \Leftrightarrow (y_0, y_0) \in U \Leftrightarrow y_0 \notin A$↯

□

Definition 2.13 (Separation) $\Gamma(X)$ as always. We say that $\Gamma(X)$ has the separation property if for every $A, B \in \Gamma(X)$ with $A \cap B = \emptyset$ there ist $C \in \Delta(X)$ such that $A \subset C$ and $C \cap B = \emptyset$

$A = \{\vec{x} \in {}^\omega\mathbb{R} \mid x_n \to 0\}, B = \to 1 \in \underset{\sim}{\Pi_3}^0({}^\omega\mathbb{R})$

$\vec{x} \in A \cup B \qquad \vec{x} \in A$ iff $\exists N \in \omega \forall n \geq N\, |x_n| \leq \frac{1}{2}$

$\Rightarrow \in \underset{\sim}{\Sigma_2}^0$

Definition 2.14 X polish, $\Gamma(X) \subset \mathcal{P}(X)$

- $\Gamma(X)$ has the *separation property* if for every $A, B \in \Gamma(X)$ with $A \cap B = \emptyset$ there ist $C \in \Delta(X)$ such that $A \subset C$ and $C \cap B = \emptyset$
- Γ has the *generalized separation property* if for every sequence $A_n \in \Gamma(X)$ with $\bigcap\limits_n A_n = \emptyset$ then there are $B_n \in \Delta(X)$ with $A_n \subset B_n$ and $\bigcap\limits_n B_n = \emptyset$
- Γ has the *reduction property* if $\forall A, B \in \Gamma(X)$ there are $A^*, B^* \in \Gamma(X)$ such that $A^* \subset A, B^* \subset B, A^* \cup B^* = A \cup B, A^* \cap B^* = \emptyset$
- Γ has the *generalized reduction property* if for every sequence $A_n \in \Gamma(X)$ there are $A_n^* \in \Gamma(X)$ such that $A_n^* \subset A_n, A_n^* \cap A_m^* = \emptyset$ if $n \neq m, \bigcup\limits_n A_n = \bigcup\limits_n A_n^*$
- Γ has the *uniformization property* if for every $P \in \Gamma(X \times Y)$ there is $P^* \subset P, P^* \in \Gamma(X \times Y)$ such that $\forall x \in X (\exists y : (x,y) \in P) \Leftrightarrow \exists! y : (x,y) \in P^*$
- Γ has the *number uniformization property* if the uniformization property is true with $Y = \omega$

Definition 2.15 Γ is *reasonable* if for every sequence $A_n \in \Gamma(X)$ the set $A = \{(x,n) \in X \times \omega \mid x \in A_n\} \in \Gamma(X \times \omega)$ and reverse

Proposition

1. If Γ has the reduction property, then $\breve{\Gamma} = \{X \setminus A, A \in \Gamma(X)\}$ has the separation

property

2. If Γ is closed under countable unions and has the generalized reduction property, then $\breve{\Gamma}$ has the generalized separation property
3. If Γ is reasonable, then Γ has the generalized reduction property iff Γ has the number uniformization property
4. If Γ is closed under continuous preimages and there is a ${}^{\omega}2$-universal set for $\Gamma({}^{\omega}2)$, then Γ cannot have both the reduction and separation property

$\underset{\sim}{\Sigma_\xi}^0$ has the number uniformization property and is reasonable

Theorem 2.12

- For every $\xi > 1$, $\underset{\sim}{\Sigma_\xi}^0(X)$ has the number uniformization property
- If X has a basis consisting of clopen sets, e.g. $X = {}^{\omega}\omega$ or $X = {}^{\omega}2$, then $\underset{\sim}{\Sigma_1}^0$ has the number uniformization property

X polish, ... $\supset A_3 \supset A_2 \supset A_1 \supset A_0, A_i \in \underset{\sim}{\Sigma_1}^0$ If $\alpha < \omega_1$, then α can be uniqueliy written as $\lambda + \nu$, where λ is limit and $\nu \in \omega$

Definition 2.16

- $\alpha < \omega_1$ is *even* (resp. *odd*) if ν is even (resp. odd) for ν such that $\alpha = \lambda + \nu$ for λ limit.
- $\alpha, \beta < \omega_2, \alpha$ and β have the same parity ($\alpha \equiv \beta$) if they are either both even or both odd

Let $\theta < \omega_1$ and let $(A_\eta)_{\eta<\theta}, A_\eta \subset X (\forall \eta < \theta)$ and $A_\eta < A_{\eta'}$ for $\eta < \eta' < \theta$

We define $D_\theta((A_\eta)_{\eta<\theta}) = \left\{ x \in X \mid x \in \bigcup_{\eta<\theta} A_\eta \wedge \text{least } \eta \text{ s.t. } x \in A_\eta \text{ is s.t. } \eta \not\equiv \theta \right\} \subset X$

For $1 \leq \xi, \theta < \omega_1$, let $D_\theta(\underset{\sim}{\Sigma_\xi}^0)(X) = \left\{ D_\theta((A_\eta)_{\eta<\theta}) \mid A_\eta \in \underset{\sim}{\Sigma_\xi}^0(X), \eta < \theta \right\}$

Let $\xi = 1$

- $\theta = 1 : D_\theta(\underset{\sim}{\Sigma_1}^0(X) = \underset{\sim}{\Sigma_\xi}^0(X)$
- $\theta = 2 : A_0 \subset A_1, A_i \in \underset{\sim}{\Sigma_\xi}^0 . D_2((A_\eta)_{\eta<\theta}) = \left\{ x \in X \mid x \in \bigcup_{\eta<2} (\underset{\sim}{\Sigma_\xi}^0)(X) \wedge \text{least } \eta : \eta \equiv 2 \right\} =$ $A_1 \setminus A_0$

- $\theta = 3 : A_0 \subset A_1 \subset A_2, D_3((A_\eta)_{\eta<\theta}) = (A_2 \setminus A_1) \cup A_0$

(Hierarchy)

Theorem 2.13 (Hausdorff/Kuratowski) Let $1 \leq \xi < \omega_1$, X polish.

$$\bigcup_{1\leq\theta<\omega_1} D_\theta(\underset{\sim}{\Sigma_\xi}^0)(X) = \underset{\sim}{\Delta_{\xi+1}}^0(X)$$

Definition 2.17 X polish space. A set $A \subset X$ ist *analytic* if either $A = \emptyset$ or there is a continuous surjection $f : {}^\omega\omega \to A$

- $\underset{\sim}{\Sigma_1}^1(X) = \{A \subset X \mid A \text{ is analytic}\}$
- $\underset{\sim}{\Pi_1}^1(X) = \breve{\underset{\sim}{\Sigma_1}}^1(X) = \left\{X \setminus A \mid A \in \underset{\sim}{\Sigma_1}^1(X)\right\}$
- $\underset{\sim}{\Delta_1}^1(X) = \underset{\sim}{\Sigma_1}^1(X) \cap \underset{\sim}{\Pi_1}^1(X)$

Theorem 2.14 X polish and uncountable, then $\mathrm{Bor}(X) \subsetneq \underset{\sim}{\Sigma_1}^1(X), \underset{\sim}{\Pi_1}^1(X), \underset{\sim}{\Delta_1}^1(X)$

Theorem 2.15 X polish and non-empty, then there is a continuous surjection $f : {}^\omega \to X$

Theorem 2.16 Let X be polish, $A, B \in \underset{\sim}{\Sigma_1}^1(X)$ s.t. $A \cap B = \emptyset$. Then there is $C \in \mathrm{Bor}(X)$ separating A from B $(A \subset C, C \cap B = \emptyset)$

Theorem 2.17 X polish $\Rightarrow \mathrm{Bor}(X) = \underset{\sim}{\Delta_1}^1(X)$

Games, strategy... (siehe skript Gloede)
A Game $G(A)$ is *determined* if either player I oder II has a winning strategy
Let $G(A)$ be the game, where I and II alternate in calling a natural number, producing an infinite sequence (x_i). I wins iff $(x_i) \in A \subset {}^\omega\omega$ Under the axiom of choice, there are $A \subset {}^\omega\omega$ s.t. $G(A)$ is not determined.

Theorem 2.18 If A is closed or open, then $G(A)$ is determined

Theorem 2.19 If A is Borel, then $G(A)$ is determined

Steel-Martin If there are enough large cardinals (infinitely many Wodin cardinaly with a measurable above) then for every projective set $A \in \bigcup_n \underset{\sim}{\Sigma_n}^1(X)$, $G(A)$ is determined

Let $\mathfrak{F}$ be a collection of functions from ${}^\omega\omega$ into itself. Let $A, B \subset {}^\omega\omega : A \leq_{\mathfrak{F}} B$ iff there is $f \in \mathfrak{F}$ s.t. $x \in A \Leftrightarrow f(x) \in B$
If $\mathfrak{F}$ contains the identity and is closed under composition, then $\leq_{\mathfrak{F}}$ is a pre-order, i.e.: reflexive and transitive
For $A, B \subset {}^\omega\omega$ we set $A \equiv_{\mathfrak{F}} B$ iff $A \leq_{\mathfrak{F}} B$ and $B \leq_{\mathfrak{F}} A$. If $\leq_{\mathfrak{F}}$ is a pre-order, then $\equiv_{\mathfrak{F}}$ is an equivalence relation on $\mathcal{P}({}^\omega\omega)$
Given $A \subset {}^\omega\omega$, the $\mathfrak{F}$-degree of A is $[A]_{\mathfrak{F}}$
The relation $\leq_{\mathfrak{F}}$ induces an order on $\{[A]_{\mathfrak{F}} \mid A \subset {}^\omega\omega\}$ - $[A]_{\mathfrak{F}} \leq [B]_{\mathfrak{F}} \Leftrightarrow A \leq_{\mathfrak{F}} B$

$A \subset {}^\omega\omega$ is $\mathfrak{F}$-*selfdual* iff $A \leq_{\mathfrak{F}} \neg A = {}^\omega\omega \setminus A$
When $\mathfrak{F} = \{f : {}^\omega\omega \to {}^\omega\omega \mid f \text{ is continuous}\}$, we notate $\leq_W$(Wadge) instead of $\leq_{\mathfrak{F}}$
When $\mathfrak{F} = \{f : {}^\omega\omega \to {}^\omega\omega \mid f \text{ is Lipschitz with constant } \leq 1\}$, we notate $\leq_L$ instead of $\leq_{\mathfrak{F}}$
$\leq_W$ and $\leq_L$ are both pre-orders. Hence: Wadge/Lipschitz equivalence: $\equiv_W / \equiv_L$ and degrees $[A]_W / [A]_L$
If $A \in$Bor$({}^\omega\omega)$, then $[A]_W \in$Bor$({}^\omega\omega)$ ($B \leq_W A$ iff $B = f^{-1}(A)$ for some continuous f)
If $A \leq_L B \Rightarrow A \leq_W B$ hence $[A]_L \subset [A]_W$

The *Wadge hierarchy* is the structure of $\{[A]_W \mid A \subset {}^\omega\omega\}$ under $\leq$ (the relation induced by $\leq_W$)
The *Borel-Wadge hierarchy* is the restriction of the Wadge hierarchy to Wadge degrees of Borel sets. (Analogous *(Borel-)Lipschitz hierarchy*)

Lipschitz-Game $A, B \subset {}^\omega\omega$ - similar to $G(A)$: In $G_L(A, B)$ two players call natural numbers creating two sequences $a, b \in {}^\omega\omega$. II wins iff $a \in A \Leftrightarrow B \in B$

Lemma 2.3 For every $A, B \in {}^\omega\omega$, $A \leq_L B$ iff II has a winning strategy in $G_L(A, B)$